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DUALITY IN DISCRETE PROGRAMMING
AND APPLICATIONS TO CAPITAL BUDGETING

A THESIS

Presented to the
Faculty of the Graduate Division
by
Peter Georg Kastil

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
in the School of Industrial Engineering

Georgia Institute of Technology

August, 1970

DUALITY IN DISCRETE PROGRAMMING
AND APPLICATIONS TO CAPITAL BUDGETING

Approved:

Chairman

Date approved by Chairman: 6/30/71

ACKNOWLEDGMENTS

I gratefully acknowledge the assistance, counsel, and guidance of my advisor, Dr. V. E. Unger. Without his help, valuable suggestions, and deep insight, this research would not have been possible, and I would like to express my appreciation to him.

I also thank the members of my reading committee, Dr. L. A. Johnson, and Dr. J. J. Jarvis for reading the original draft of this thesis and being helpful by their suggestions and criticism.

Special thanks are due to Dr. C. M. Shetty. Dr. Shetty has helped the author throughout the study by his valuable suggestions. He has always been approachable, and his mathematical ability has been very valuable.

I also would like to thank Dr. W. W. Hines, Associate Director, Graduate Programs, School of Industrial and Systems Engineering, Georgia Institute of Technology. Dr. Hines has guided me throughout my study and his interest and assistance I greatly appreciate.

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SUMMARY

A duality concept for discrete programming due to Balas is introduced for the case of two "dual" problems with a Lagrangian-type objective function, where the min-max/max-min of a linear function is to be found over a domain defined by linear inequalities. The variables are constrained to belong to arbitrary sets of real numbers, i.e. some or all of the variables may be discrete. After a discussion of the consequences of including indivisibilities in linear economic models for the application of shadow-price systems and decentralized decision making procedures, economic interpretations are presented for the case of a mixed-integer programming problem, whose dual has been formulated following Balas's duality concepts. A generalized shadow-price system, involving nonnegative dual variables and unrestricted subsidies and penalties, is introduced.

Balas's duality concept and its economic implications then are applied to four specific capital budgeting models. The first three models are based on formulations by Weingartner and Baumol and Quandt; they include, however, the additional assumption of indivisible projects and lead to new results in the interpretation of their optimal solutions. A finite decentralized decision making procedure is one of these results. By combining and extending the mentioned capital budgeting models, a formulation is obtained, the solution of which yields the optimal investment, dividend, and financing policy of a firm interacting with an imperfect capital market.

CHAPTER I

INTRODUCTION

Among the most important decisions which must be made in any company are those concerning capital investment. These decisions, which are made by top management, involve large sums of money and therefore usually require extensive planning, budgeting, and funding activities. In general, both external factors, over which the management of the company has minimal control, such as competition and economic environment, and internal factors, as budgeting practices, long- and short-term objectives, must be considered.

In order to make correct capital expenditure decisions, corporate management needs at least three sets of information. Estimates must be made of net capital outlays required and future cash earnings promised by each proposed project. This is a problem of engineering valuation and market forecasting. Estimates must be made of the availability and cost of capital to the company. This is a problem of financial analysis. Finally, management needs a correct set of standards by which to select projects for execution so that long-run economic benefits to present owners will be maximized. This is basically a problem in logic and applied mathematics. In this research, we shall be concerned with the last two of the mentioned problems.

Investment criteria have been dealt with extensively in the business and economic literature. Based on the well-known conventional

concepts of present worth, internal rate of return, risk preference, and utility functions, operations research methodology has been applied to construct and solve mathematical models for the capital budgeting problem.

Since Weingartner's important contribution for the correct theoretical formulation and treatment of constrained capital budgeting problems in 1963 (15) linear and integer programming techniques have been used to develop systematic approaches to these problems; and, in the case of linear programming models, the concept of duality has provided interesting relationships for clarifying and interpreting many aspects of capital budgeting which have not been effectively treated by the conventional methods of economic theory.

These successful interpretations, however, have been based on the assumption of continuity, i.e. that all quantities (variables and constants) in the model considered can be measured by real numbers. In the more realistic cases, where indivisibilities occur, all-integer and mixed-integer programming techniques must be used which have considerable drawbacks, both in terms of computations required and, in particular, in the interpretation of the solutions. Several attempts to construct a system of "dual shadow prices" as in linear programming models have been made without yielding completely satisfactory results.

In this research, a generalized shadow-price system based on a duality construction by Balas (2) is introduced that avoids the usual problems encountered for pricing schemes in all-integer and mixed-integer programming. This approach is based on a theorem that allows the formulation of "equivalent linear programs" to mixed-integer or pure-integer

programs. The result of such a construction is a system of non-negative dual shadow-prices and unconstrained quantities which are shown to have the meaning of subsidies and penalties for the profit characteristics of the various activities considered.

The first part of this thesis deals with a discussion and analysis of Balas's duality for the linear case. Balas has studied a pair of dual problems in which the min-max/max/min of a linear function is to be found over a domain defined by linear inequalities, and some of the variables are constrained to belong to arbitrary sets of real numbers. Mixed-integer and hence all-integer (linear) programs are shown to be special cases of these problems. A number of properties for this pair of dual problems can be stated which contain, among others, symmetry, existence, complementary slackness, and uniqueness relations.

This section is followed by a discussion of the problems arising in economic analysis when indivisibilities are included, and the introduction to the concepts of shadow-prices, marginal and opportunity costs, competitive markets, and others. These concepts are first presented in a linear programming framework, whereby, as an example, the model of a production firm in a purely competitive market is chosen. The relevant aspects of centralization and de-centralization are discussed, and then a generalized shadow-price system for discrete programming problems is introduced.

The second part of this research is devoted to the application of Balas's duality construction and its economic implications to four specific capital budgeting models. The first three models are based on formulations by Weingartner and Baumol and Quandt; they include, however,

the additional assumption of indivisible projects and lead to new results in their economic interpretation. In each case, the aspects of decentralization are considered, and for the "Terminal Wealth Model", a procedure for decentralized decision making, based on Benders' Partitioning Algorithm, is developed. Finally, combining these models and extending them, a formulation is found which includes both dividend payments and terminal wealth of the firm in its objective function, and whose constraints contain upper limits on borrowing.

CHAPTER II

BALAS'S DUALITY CONCEPT IN DISCRETE PROGRAMMING

AND ECONOMIC INTERPRETATIONS

2.1 Introduction

In mathematical programming, duality theory has been of considerable importance and a number of dual problems for linear and nonlinear primal problems have been formulated, under various assumptions on the structure of the primal. In linear programming, duality principles are readily established using the theory of linear inequalities (11). The main result of linear programming duality theory is that the primal problem has an optimal solution if and only if the dual has one, in which case the values of both objective functions are equal at optimality.

In nonlinear programming it turns out that, in order to establish a symmetrical duality, rather strong conditions have to be imposed on the problem functions, while, under less stringent conditions, a "one-way duality" can be established: starting with one problem, under certain conditions, a solution to this problem provides a solution of the other problem. Customarily, one speaks of a dual relationship between such a pair of problems, even if the fully symmetrical duality properties do not hold.

Balas (2) has studied a pair of dual problems, in which the min-max (max-min) of a linear function is to be found over a domain defined by linear inequalities, and the variables are constrained to belong to arbitrary sets of real numbers, e.g. some of all of the

variables may be discrete. Mixed-integer, and hence all-integer (linear) programs are special cases of these problems. Balas's duality construction is symmetric, i.e. the dual of the dual is the primal. Subject to a qualification, the primal has an optimal solution if and only if the dual has one, and in this case, equality between the values of the respective objective functions occurs.

Additional properties of the pair of dual problems considered by Balas include conditions for the existence of feasible and finite optimal solutions, uniqueness of the optimum, and the relationship between the dual of a mixed-integer program and the dual of the linear program which is defined over the convex hull of feasible points to the mixed-integer program.

Balas has extended the above results to the case of a quadratic objective function with negative, semi-definite quadratic forms (3), as well as to the case of a non-linear objective function and nonlinear constraints (4). Again, some of the variables are constrained to belong to arbitrary sets of real numbers.

The basic properties established for nonlinear programs with exclusively continuous variables can be shown to carry over, with some qualification, to their generalized partly discrete counterparts that contain, among others, mixed-integer nonlinear programs.

The first part of this chapter will be devoted to a discussion and analysis of Balas's duality concept. We will restrict ourselves to the linear case, which is of great relevance for the construction of the capital budgeting models in later chapters, and will state several theorems that describe the nature of this duality. The second part of

the chapter, then, will contain several economic interpretations of results from mathematical programming theory in general, and Balas's duality in particular.

2.2 Formulation of Primal and Dual

Problems in the Linear Case

Consider the pair of dual linear programs,

$$\begin{array}{ll}
 \text{max } cx & \text{min } ub \\
 \text{(LP) subject to} & \text{(LD) subject to} \\
 Ax + y = b & uA - v = c \\
 x, y \geq 0 & u, v \geq 0,
 \end{array}$$

where A is a $m \times n$ matrix, c , x , and v are n -component vectors, and b , y , and u are m -component vectors. Two index sets are defined by $M \equiv (1, \dots, m)$, and $N \equiv (1, \dots, n)$.

The main result of linear programming duality theory is that the primal problem has an optimal solution if and only if the dual has one, in which case, denoting the two optimal solutions by (\bar{x}, \bar{y}) and (\bar{u}, \bar{v}) respectively, we have $\bar{c}\bar{x} = \bar{u}\bar{b}$, and $\bar{u}\bar{y} = \bar{v}\bar{x} = 0$. These relations play a central role in linear programming.

Suppose now, the first n_1 components of x and the first m_1 components of u ($0 \leq n_1 \leq n$, $0 \leq m_1 \leq m$) are arbitrarily constrained, and the following notation is introduced:

$$\begin{aligned}
 (x_1, \dots, x_{n_1}) &= x^1, (u_1, \dots, u_{m_1}) = u^1, x = (x^1, x^2), \\
 u &= (u^1, u^2), (1, \dots, n_1) = N_1, (1, \dots, m_1) = M_1.
 \end{aligned}$$

Let us further partition A , b , c , y , and v in accordance with the partitioning of x and u :

$$A = \begin{pmatrix} \underbrace{A^{11}}_{N_1} & \underbrace{A^{12}}_{N_2} \\ \underbrace{A^{21}}_{N_1} & \underbrace{A^{22}}_{N_2} \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}} \right\} M_1 \\ \left. \vphantom{\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}} \right\} M_2 \end{matrix} \quad M$$

$$b = (b^1, b^2), \quad c = (c^1, c^2)$$

$$y = (y^1, y^2), \quad v = (v^1, v^2)$$

N

Then Balas (2) considers the following general optimization problem:

$$\begin{aligned} & \min_{u^1} \max_x cx + u^1 y^1 + u^1 A^{11} x^1 \\ (P) \quad & \text{subject to} \end{aligned}$$

$$Ax + y = b$$

$$x^1 \in X^1, \quad u^1 \in U^1$$

$$x^2, y^2 \geq 0$$

$$y^1 \text{ unconstrained.}$$

X^1 and U^1 are arbitrary sets of vectors in the n^1 -dimensional and m^1 -dimensional Euclidean space, i.e. some of the components of the vectors x and u are arbitrarily constrained. The fact that all slack variables belonging to y^1 are unconstrained indicates a "partial relaxation" of the constraint equations in (P).

Balas defines the following problem to be the dual to (P):

$$\begin{aligned} & \max_{x^1} \min_u \quad ub - v^1 x^1 + u^1 A^{11} x^1 \\ (D) \quad & \text{subject to} \end{aligned}$$

$$uA - v = c$$

$$u^1 \in U^1, \quad x^1 \in X^1$$

$$u^2, v^2 \geq 0$$

$$v^1 \text{ unconstrained.}$$

Since, in the above pair of problems, y is uniquely defined by x , and v is uniquely defined by u , a solution to (P) will be written as (x, u^1) , and a solution to (D) as (u, x^1) .

Rewriting (D) in the primal form (i.e. in the form of (P)), we obtain (by changing the signs in the objective function and in the equation set)

$$\begin{aligned} & \min_{x^1} \max_u \quad u(-b) + v^1 x^1 + u^1 (-A^{11}) x^1 \\ & \quad u(-A) + v = (-c) \\ & \quad u^1 \in U^1, \quad x^1 \in X^1, \quad v^1 \text{ unconstrained} \\ & \quad u^2, v^2 \geq 0. \end{aligned}$$

The formulation of the dual to this problem yields (P). Thus, the duality defined in this way is involutory (symmetric). Balas states this property in a theorem.

Theorem 1 (Involution): The dual of the dual is the primal.

The main feature of the above pair of dual problems is the special relationship between each primal variable x_j and the associated dual slack v_j , and between each dual variable u_i and the associated primal slack y_i , namely

$$\begin{aligned}
 (2.1) \quad & x_j \text{ arbitrarily constrained} \longleftrightarrow v_j \text{ unconstrained} \\
 & x_j = 0 \longleftrightarrow v_j = 0 \\
 & y_i \text{ unconstrained} \longleftrightarrow u_i \text{ arbitrarily constrained} \\
 & y_i = 0 \longleftrightarrow u_i = 0.
 \end{aligned}$$

If we now consider the case where the arbitrary constraints of the form $x^1 \in X^1$, $u^1 \in U^1$ are integrality constraints on the components of x^1 and u^1 , then our pair of dual problems subsumes the following special cases:

(a) If $M_1 = \emptyset$, then P becomes a mixed-integer linear program I. Its dual D is then a constrained mixed-integer optimization problem (max-min) of a special type (nonnegativity being the sole constraint on the integer variables).

(b) If $M_1 = \emptyset$ and $N - N_1 = \emptyset$, then P is a pure integer programming problem. The dual of such a problem turns out to be a mixed-integer optimization problem (max-min) over the nonnegative orthant, otherwise unconstrained.

(c) If $M_1 = \emptyset$ and $N_1 = \emptyset$, P and D become a pair of dual linear programs.

(d) If $M - M_1 = \emptyset$ and $N - N_1 = \emptyset$, then both P and D are pure integer optimization problems (min-max and max-min respectively) over the nonnegative orthant, otherwise unconstrained.

Of course, $N_1 = \emptyset$ gives rise to the converse of (a), $N_1 = \emptyset$ and $M - M_1 = \emptyset$ to the converse of (b). Several of the special cases listed above will be considered in greater detail in the subsequent chapters.

We now denote all x verifying the constraints of P with X , and all u verifying the constraints of D with U so that the Cartesian Product $X \times U^1$ is the set of feasible solutions (x, u^1) to P , and $U \times X^1$ is the set of feasible solutions (u, x^1) to D .

We further let

$$z = \min_{u^1 \in U^1} \max_{x \in X} cx + u^1 y^1 + u^1 A^{11} x^1,$$

and

$$w = \max_{x^1 \in X^1} \min_{u \in U} ub - v^1 x^1 + u^1 A^{11} x^1.$$

Before stating the next theorem, we need the following two definitions. Let s^1, \dots, s^p be elements of arbitrary vector spaces.

A vector function $G(s^1, s^2, \dots, s^p)$ will be called separable with respect to s^1 if there exist vector functions $H(s^1)$ (independent of s^2, \dots, s^p), and $K(s^2, \dots, s^p)$ (independent of s^1), such that

$$G(s^1, s^2, \dots, s^p) \equiv H(s^1) - K(s^2, \dots, s^p).$$

$G(s^1, s^2, \dots, s^p)$ will be called componentwise separable with respect to s^1 , if each component g_i of G can be written either as $g_i(s^1)$, or as $g_i(s^2, \dots, s^p)$.

Using this concept of separability, Balas has stated and proved the following theorem.

Theorem 2 (Saddle-Point). Assume that v^2 (or y^2) is componentwise separable with respect to u^1 (to x^1). Then, if P has an optimal solution (\bar{x}, \bar{u}^1) , there exists \bar{u}^2 such that (\bar{u}, \bar{x}^1) , where $\bar{u} = (\bar{u}^1, \bar{u}^2)$, is an optimal solution to D, with

$$\begin{aligned} \min_{u^1 \in U^1} \quad & \max_{x \in X} \quad cx + u^1 y^1 + u^1 A^{11} x^1 \\ & = \max_{x^1 \in X^1} \quad \min_{u \in U} \quad ub + v^1 x^1 - u^1 A^{11} x^1, \end{aligned}$$

and the function

$$F(x, u) = cx + ub - uAx + u^1 A^{11} x^1$$

has a saddle-point at (\bar{x}, \bar{u}) :

$$F(x, \bar{u}) \leq F(\bar{x}, \bar{u}) \leq F(\bar{x}, u)$$

for all $x \in X(\bar{y}^2)$, $u \in U(\bar{v}^2)$. These sets are defined by

$$X(\bar{y}^2) = \left\{ x \in X \mid A^{21} x^1 + A^{22} x^2 = b^2 - \bar{y}^2 \right\},$$

$$U(\bar{v}^2) = \left\{ u \in U \mid u^1 A^{12} + u^2 A^{22} = c^2 + \bar{v}^2 \right\}.$$

Let us now denote

$$X^+ = \left\{ x \in X \mid A^{12} x^2 \leq b^1 \right\},$$

$$U^+ = \left\{ u \in U \mid u^2 A^{21} \leq c^1 \right\}.$$

Then Balas's next theorem reads:

Theorem 3 (Existence). If v^2 (or y^2) is componentwise separable with respect to u^1 (to x^1), then exactly one of the following five situations holds for P and D:

- (1) $z = +\infty$, $w = +\infty$ and $U^* = \emptyset$
- (2) $z = -\infty$, $w = -\infty$ and $X^* = \emptyset$
- (3) z and w are undefined; $X^* = \emptyset$, $U^* = \emptyset$
- (4) $z = -\infty$, $w = +\infty$, $X = \emptyset$, $U = \emptyset$
- (5) z and w are finite, and $z = w$.

Balas extends the complementary slackness concept of linear programming to the more general case of the dual problems P and D.

Theorem 4 (Complementary Slackness). If (\bar{x}, \bar{u}^1) and (\bar{u}, \bar{x}^1) are optimal solutions to P and D respectively, then

$$\bar{u}^2 y^2 = 0, \quad \bar{v}^2 \bar{x}^2 = 0,$$

and

$$(c^2 - \bar{u}^1 A^{12}) \bar{x}^2 - \bar{u}^2 (b^2 - A^{21} \bar{x}^1) = 0.$$

Balas furthermore establishes a uniqueness relationship between the optimal solutions to P and D.

Definition. $(x, u^1) \in X \times U^1$ is called an extreme solution to P, if x is an extreme point of the convex hull of X .

It is obvious that if P has an optimal solution, then it has an optimal extreme solution.

Definition. Let (x, u^1) be an extreme solution to P; let p and q be the number of positive components of x^2 and y^2 , respectively. Then (x, u^1) is called a nondegenerate (degenerate) solution to P if

$$p + q = m - m_1 \text{ (if } p + q < m - m_1 \text{) } .$$

Theorem 5 (Uniqueness). Let (\bar{x}, \bar{u}^1) be a unique, nondegenerate optimal solution to P. Then D has a unique optimal solution (\bar{u}, \bar{x}^1) .

The proofs to the above theorems are contained in Balas's paper (2).

2.3 Economic Interpretations

Indivisible Resources

One of the usual assumptions in economic theory is that resources or commodities can be measured by real numbers. Production functions, demand curves, and cost functions are assumed to be defined for real number arrays and to behave properly with respect to various criteria of continuity. Assumptions of this sort simplify economic analysis. They imply, however, an acceptance of commodity divisibility and ignore the difficulty or even impossibility, in practice, of using or producing fractional units of a commodity. In many instances, indivisible rather than divisible commodities (a "commodity" here may be either a service or a good) are the more realistic to deal with.

Two possible reasons, why writers in economic theory rarely include indivisibilities in their analysis, might be the following: first, the tools of algebra and mathematical analysis are much less powerful in dealing with discrete quantities (variables and constants), whereas they are most useful in the case of continuous concave functions and convex sets. In analyzing indivisibilities, the types of functions with which one is concerned are usually discrete and neither concave nor convex; also, the sets on which these functions are defined are not usually

convex. Secondly, many writers argue that indivisible commodities would make no significant difference in their analysis. Either divisibility is assumed to be a sufficient approximation to the real world phenomena with which they are concerned or the analysis would be identical or at least similar if commodities were assumed to be discrete.

Frank (9), who has given a detailed discussion of this topic in a recent work, points out that the "effects of indivisibilities" tend to be finite regardless of scale. For a sufficiently large scale, the effects of indivisibilities are "averaged out," and their relevance is not so great. Sometimes, one might justifiably ignore them, but there are several important areas of economic analysis in which progress depends on the development of methods for solving or analyzing problems in the efficient allocation of indivisible resources. There are, for instance, practical decision problems such as determining suitable numbers of machine tools of various kinds within a plant, or choosing the number and sizes of dams in river valley development. In later chapters, we will deal with indivisible investment opportunities, where the financial decision maker of a firm is faced with the problem of selecting from a set of available investment projects, those that maximize a predetermined goal of the firm and satisfy various possible restrictions on budget, material, and other resources. Furthermore, indivisibilities are in many cases at the root of increasing returns to scale, whether arising within a plant or firm, or in relation to a number of firms through "external economies."

Pricing Concepts in the Theory of the Firm

This section will be devoted to the introduction to the important concepts of shadow prices, marginal prices, competitive markets, opportunity costs, and others. These concepts will first be presented in a linear programming framework and then will be extended to the case of mixed or pure discrete programming models.

Let us consider a firm that produces goods in various processes of production, and let m be the number of factors of production (resources) whose supply is limited. For a given process of production, let a_{ij} units of resource i be required to produce one unit of good j . Then, the so-called "activity vector", $a_j = (a_{1j}, \dots, a_{mj})$, tells how much of each resource is required to make one unit of good j .

If we produce x_j units of good j (j refers to a given good produced by a given production process), the variable x_j is called the "level" at which activity j operates. Let further b_i be the maximum amount of resource i available in the time period under consideration, and let c_j be the profit on one unit of the good made by activity j . We now define our problem as the task of determining the levels at which the activities should be operated to maximize the firm's profit rate.

To construct our linear programming model we use the further assumption that the firm sells its products in a purely competitive market. Pure competition implies that a large number of firms make the same product, and no firm can influence the market price. Therefore, since our approach is essentially static, and since no interaction exists between producers and consumers, the maximization of the profit

rate at each instant of time is equivalent to maximizing the total profit over all time.

Under the assumptions stated above, we can write our problem algebraically in the form

$$\begin{array}{ll}
 & \max cx \\
 \text{(LP)} & \text{subject to} \\
 & Ax \leq b \\
 & x \geq 0,
 \end{array}$$

where c and x are n -vectors consisting of elements c_j and x_j respectively; b is a m -vector with elements b_i ; and A is a $m \times n$ matrix with elements a_{ij} .

The physical dimensions of the variable x_j are the units of some goods produced for a given time period. The dimensions of b_i are units of resource i available in the given time period, the dimensions of the c_j are dollars per unit of good j .

Consider now the dual problem of (LP)

$$\begin{array}{ll}
 & \min ub \\
 \text{(LD)} & \text{subject to} \\
 & uA \geq c \\
 & u \geq 0,
 \end{array}$$

where u is a m -vector with elements u_i . The dimensions of the dual variable u_i are dollars per unit of resource i .

To each resource i there corresponds a dual variable u_i which, by its dimensions, is a price, or cost, or value to be associated with one unit of resource i . Thus, ub is the total value of the available

resources. The j -th constraint of (LD) reads $\sum_{i=1}^m a_{ij}u_i \equiv c_j$, where the left-hand side of this inequality is the value of the resources used in making one unit of product j . The dual problem determines the u_i so that the total value of the resources is minimized, and the value of the resources used in producing one unit of j is at least as great as the profit received from selling one unit of j .

The dual variables u_i are referred to as dual variables, shadow prices, or imputed values for the resources. Note that they have nothing to do with the actual costs of the resources; the c_j are profits, and thus the actual costs of the resources never appear. Instead, the dual variables may be considered as evaluators of the resources, which, in a certain sense, provide a way of measuring the contribution of each resource i to the profit, cx .

Be complementary slackness, for $x_j > 0$ in the optimal solution, $\sum_{i=1}^m a_{ij}u_i = c_j$, so that for the activities used, the value of the resources used to produce one unit of j is precisely equal to the profit. If the i -th primal constraint is a strict inequality (the corresponding slack variable is positive), so that not all of resource i is used, then $u_i = 0$, and the cost or value of that resource is 0. We call such a resource a "free good."

The valuation of the resources by means of the u_i is an opportunity cost valuation. This becomes evident by considering the following facts. At optimality, $cx = ub$, i.e. the maximum profit is equal to the minimum value of the resources. Now, if it were possible to increase or decrease the amount available of resource i by one unit, without changing the solution to the dual, the maximum profit

would be increased or decreased by u_i , i.e. here is the basis for an opportunity cost interpretation; of course, if we actually change b_i to $b_i + 1$, the profit will not necessarily increase by u_i because the optimal basis may change. However, when cx is being maximized, u_i is a measure of the rate of change of cx with respect to b_i , until the basis changes. When a resource is not fully utilized in an optimal solution cx will not change if the availability of the resource is increased indefinitely. Hence, u_i for this resource should be zero, which is in fact ensured by the complementary slackness conditions. The dual variables as presented here have applications in various areas, and their interpretation is one of our major concerns in the following chapters.

Decentralized Decision Making

The dual variables u_i have potential applications in cost accounting. Consider a large decentralized corporation which is broken down into a number of departments. Each department may have several products. In addition, there may exist a number of different activities for making a single product. These various departments jointly use manufacturing facilities and other services or resources which are in limited supply. Suppose that the chief executive has obtained an optimal solution to the linear programming problem for the entire corporation, and, therefore, knows which activities should be used and what their levels should be.

Top management wishes to make sure that the department managers select the proper activities. However, this selection process should originate with the department managers and not come about as a result

of directions from top management, i.e. the decision process should be decentralized. Suppose that to each resource which is short in supply we assign the cost u_i , where u_i is the i -th dual variable in the optimal solution to the corporate problem referred to above. For each unit of resource i , a department manager must pay u_i (regardless of the actual cost of this resource). Then the cost of one unit of good i produced by activity j is $\sum_{i=1}^m a_{ij}u_i = z_j$. We now suppose that the department manager is paid c_j for each unit, where c_j is the unit profit of j . If $z_j > c_j$, the department manager will find that his department is losing money when activity j is operated at a positive level. There remains the problem of getting the managers to operate these activities at a correct level. This cannot be done by means of the above costing procedure, and hence other approaches must be used.

Dantzig and Wolfe (7) have developed a computational technique, called decomposition, that permits the solution of large and complex linear programs by solving a series of small size problems instead. In its economic interpretation, this is a procedure for decentralized decisions by a multi-division firm or a multi-sector economy. The method includes in its operations a coordinating mechanism which prevents the decentralized decisions from working at cross purposes. This mechanism employs as its instrument a generalized interpretation of the shadow prices of linear programming duality.

To summarize, the decomposition approach constitutes a decision mechanism for an entire decentralized, but coordinated economic organization. However, the degree of autonomy of the divisional decision maker should not be overstated, as pointed out by Baumol and Quandt (5).

True, the calculation process is sufficiently localized that central management does not have to know anything about the internal technological arrangements of the division. But, in the final analysis, the output decisions are made and enforced by the central planner, though based on divisional plans and proposals.

A Generalized Shadow-Price System for Discrete Programming

Balas's duality construction for discrete programming models leads in a natural way to an economic interpretation in which the shadow-price system of linear programming is replaced by a generalized shadow-price system. This generalized system consists of nonnegative dual prices, u_i , associated with each constraint i , and unconstrained subsidies or penalties s_j , associated with each discrete variable (activity) j .

To obtain the results indicated, Balas considers the pair of dual problems (dual in the sense of Balas's duality)

$$\begin{aligned}
 & \min_{u^1} \max_x \quad cx + u^1 y^1 + u^1 A^{11} x^1 \\
 (P) \quad & \text{subject to} \\
 & Ax + y = b \\
 & x, u^1 \geq 0 \\
 & x_j \quad \text{integer, } j \in N_1 \\
 & u_i \quad \text{integer, } i \in M_1 \\
 & y_i \quad \begin{cases} \text{unconstrained, } i \in M_1 \\ \geq 0, & i \in M - M_1 \end{cases}
 \end{aligned}$$

and

$$\max_{x^1} \min_u \quad ub - v^1 x^1 + u^1 A^{11} x^1$$

(D) subject to

$$uA - v = c$$

$$x_j, u_i \quad \text{integer, } j \in N_1, i \in M_1$$

$$\text{unconstrained, } j \in N_1$$

$$v_j \geq 0, \quad j \in N - N_1,$$

which were the subject of our discussion in Section 2.2.

Instead of trying now to interpret the meaning of the problem variables directly, Balas formulates, by means of a theorem, an equivalent linear program to (P) and analyzes the properties of its optimal solution in terms of the discrete program. The following theorem has been stated and proved by Balas. For clarity, we repeat the proof, in more detail.

Theorem 6. If (\bar{x}, \bar{u}^1) and (\bar{u}, \bar{x}^1) are optimal solutions to (P) and (D) respectively, then \bar{x} is an optimal solution to the linear program

$$\begin{aligned} & \max (c - s) \\ \text{(ELP)} \quad & \text{subject to} \\ & A^2 x \leq b^2 \\ & x \geq 0, \end{aligned}$$

where $A^2 = (A^{21}, A^{22})$ is the part of the matrix A that corresponds to linear dual variables u^2 , and where $s = (s_j)$ is defined as

$$(2.2) \quad s_j = \begin{cases} t_j + \bar{v}_j & \text{for } j \in N_1, \text{ such that } \bar{x}_j > 0 \\ t_j + \min(0, \bar{v}_j) & \text{for } j \in N_1, \text{ such that } \bar{x}_j = 0 \\ t_j & \text{for } j \in N - N_1, \end{cases}$$

$$\text{with } t_j = - \sum_{i \in M_1} \bar{u}_i a_{ij}, \quad j \in N.$$

Proof: \bar{x} is a feasible solution to (ELP), since the set of constraints in (ELP) is only a part of the constraint set of (P) and the additional discreteness restriction on a part of the variables are relaxed.

The dual of (ELP) can be written as

$$\begin{aligned} & \min \quad u^2 b^2 \\ (ELD) \quad & \text{subject to} \\ & u^2 A^2 \geq c + s \\ & u^2 \geq 0. \end{aligned}$$

Using the definition equation (2.2) for the subsidy vector s , and substituting into (ELD), we see that \bar{u}^2 is feasible to (ELD) and that the following relations hold for both linear programs:

$$(2.3) \quad \bar{u}^2 (b^2 - A^2 \bar{x}) = 0, \quad (c + s - \bar{u}^2 A^2) \bar{x} = 0.$$

The equations (2.3) are readily seen to be the complementary slackness conditions, what implies that \bar{x} , \bar{u}_2 are optimal to (ELP) and (ELD) respectively. Q.E.D.

This theorem relates our general maximization problem (P) and the equivalent linear program (ELP).

Let us now, in analogy to the linear model we used for the theory of the firm in the preceding chapter, consider the case when $M_1 = \emptyset$, i.e. when (P) is a mixed-integer linear program. This will now be written as

$$\begin{aligned}
 & \max \quad c^1 x^1 + c^2 x^2 \\
 \text{(MIP)} \quad & \text{subject to} \\
 & A_1 x^1 + A_2 x^2 \leq b \\
 & x^1, x^2 \geq 0 \\
 & x_j \text{ integer, } j \in N_1,
 \end{aligned}$$

where $(A_1, A_2) = A$.

In this case, if we write $s = (s^1, s^2)$, we have $s^2 = 0$, $s_j = \bar{v}_j$ for $j \in N_1$ such that $\bar{x}_j > 0$, and $s_j = \min(0, \bar{v}_j)$ for $j \in N_1$, such that $\bar{x}_j = 0$.

Thus, (ELP) becomes

$$\begin{aligned}
 & \max \quad (c^1 + s^1) x^1 + c^2 x^2 \\
 \text{(ELP)} \quad & \text{subject to} \\
 & A_1 x^1 + A_2 x^2 \leq b \\
 & x^1, x^2 \geq 0.
 \end{aligned}$$

From the results above it follows that, if (P) has an optimal solution, there exists a vector of prices $u_1 = 0$, $i \in M$, and subsidies or penalties s_j , $j \in N_1$, with the following properties:

- (a) An optimal solution x to the mixed-integer program (MIP) is an optimal solution to the linear program (ELP);
- (b) $uA_1 \leq c^1 + s^1$, $uA_2 \leq c^2$,

- (c) $(uA_1 - c^1 - s^1)x^1 = 0$, $(uA_2 - c^2)x^2 = 0$, i.e. an activity is run at a positive level only if there is no loss associated with it;
- (d) $u(b - A_1x^1 - A_2x^2) = 0$, i.e. if a price is positive, the associated resource (or commodity) is "scarce";
- (e) $(c^1 + s^1)x^1 + c^2x^2 = ub$, i.e. the total "value" of the (result of) activities is equal to the total imputed value of the resources (commodities);
- (f) if the optimal solution x to (MIP) is unique and nondegenerate (as defined in Section 2.2), then u and s^1 are unique;
- (g) some of the activities that are not run at a positive level may have to bear a penalty.

Property (a) is a special case of the result of Theorem 6; (b) represents the constraint set of the dual to (ELP); (c) and (d) are complementary slackness conditions; (e) expresses the symmetry relationship between primal and dual objective functions; (f) follows from the uniqueness property of the general pair of dual problems (P) and (D); and (g) interprets the fact that $s_j \approx 0$ for $x_j = 0$ in the optimal solution.

Since the relationships (a) through (g) are formulated for a pair of linear programs, the dual variables can be interpreted as regular shadow prices, and the concepts developed for the linear model of a profit-maximizing firm can be carried over. Here, however, this system of shadow prices is accompanied by a system of subsidies and penalties, which are used to modify the profits in the objective function of (MIP) according to the stated rule. Obviously, \bar{x} need not be (is generally not) an optimal solution to the linear program obtained from (MIP) by abandoning the integrality conditions. Figure 1 shows an example for

such a situation in two dimensions. \bar{x}_{LP} and \bar{x}_{MIP} stand for the optimal solution to the linear and the mixed-integer program respectively.

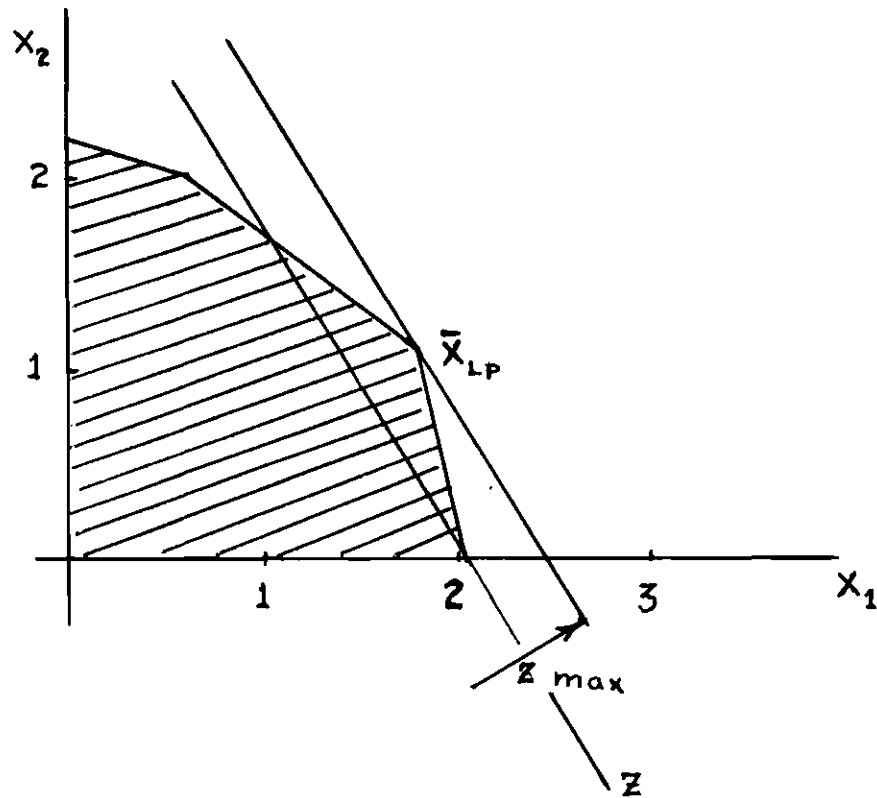


Figure 1. Two-dimensional Example with One Discrete and One Continuous Variable.

Let C be the convex polytope defined by the constraints

$$A_1 x^1 + A_2 x^2 \leq b, \quad x^1, x^2 \geq 0.$$

A feasible solution to (MIP) then must lie in C and satisfy the integrality constraints on the components of x^1 . An optimal solution \bar{x} to (MIP) lies either on a facet of C or on one of the extreme points of C .

Since the replacement of (c^1, c^2) in the objective function of (MIP) by $(c^1 + s^1, c^2)$ causes \bar{x} to become optimal with respect to this new objective function defined on C , the meaning of this replacement is that of changing the slope of the objective-function-hyperplane so as to make it parallel to the facet of C containing \bar{x} . If this facet contains more than the single point \bar{x} , then all noninteger solutions lying on it are also optimal with respect to the new objective function $(c^1 x^1 + s^1 x^1 + c^2 s^2)$, and form alternate optimal solutions to (ELP).

Although the properties (a) through (g) were stated for a pair of linear programs, the concept of a "free good" or a "free resource" has to be modified. Relation (d) does not ensure that only free resources have a zero price. In linear programming, because of the implied infinite divisibility of all inputs and all outputs, the appearance of a zero dual variable is generally, but not always, associated with a non-binding constraint in a way which suggests that the omission of the indicated constraint would not affect the optimal solution. More precisely, the indicated constraint can be omitted if and only if its associated dual variable is zero in every optimal solution.

In our case of indivisible resources for (MIP), this need not be the case. Specifically, an original constraint may not be active at the optimum to the mixed-integer program, i.e. $y_1 > 0$ and hence $u_1 = 0$, for some $i \in M$, but its omission would fail to yield the same optimal solution. Such a situation may be illustrated by means of the following diagram.

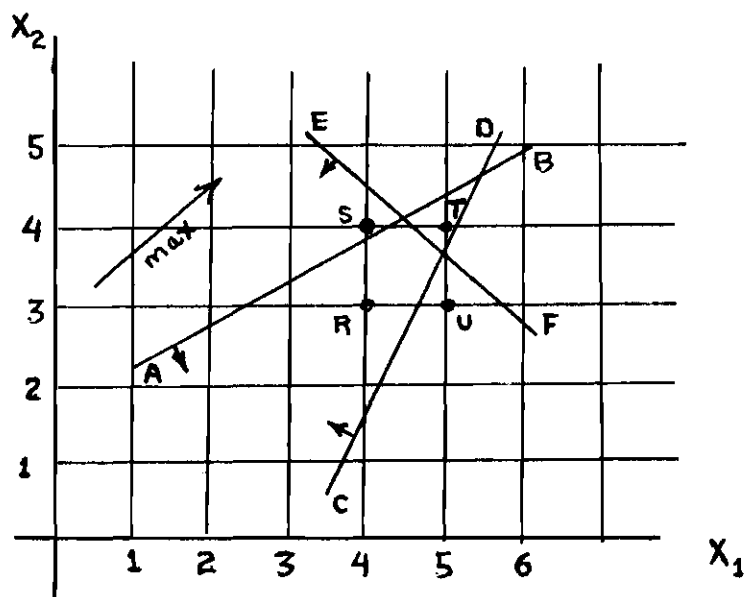


Figure 2. Case of Two Integer x_j

In Figure 2, we consider for simplicity the case of only two integer variables. The function to be maximized is assumed to increase in the direction of the arrow. Then, clearly, the optimal lattice point is R. Even though the solution space is two-dimensional, all three restrictions here are essential. If restriction EF is omitted, the optimum shifts to lattice point S; and if restriction CD is omitted, the optimum shifts to lattice point U. Although none of the restrictions in Figure 2 is actually binding at the optimal solution, they are all necessary to achieve the optimal point R. In linear programming, as already mentioned, a zero dual price is associated with a basic slack variable, i.e. a constraint with excess capacity. In integer programming, there is no assurance that a restriction is not essential even with the

appearance of the slack of a constraint in the optimal basis at a positive level. Let us define therefore a "free good" in the context of our mixed-integer programming problem as an excess resource that is not only characterized by a zero dual variable, but also by the fact that the primal constraint associated with this resource is redundant.

The generalized shadow-price system discussed is not sufficient by itself to lead to an optimal integer solution (integer in the required components) through decentralized decision making. The reason is that after the dual prices u_i have been announced by the central authority of the corporation and after the single divisions have evaluated their investment opportunities, the ranking order thus established may be altered by interference of headquarters in form of subsidies and penalties which are by-products of solving a corporate programming problem. It will be shown, however, in one of the following chapters that under certain conditions decentralization may be obtained through further analysis.

Alternate Approaches to Pricing in Integer Programming

Gomory and Baumol (12) proposed a process to obtain dual evaluators of integer programming constraints. These quantities, which they have called the "recomputed duals," assign values exclusively to the original constraints, distributing to these the duals associated with the additional constraints of the cutting planes in their algorithm. The recomputed duals are characterized by the following properties:

Their numerical values generally depend on the choice of the cutting planes, i.e. on the path utilized in the computations, and an

explicit record of the added constraints is needed for performing the computation.

They share with the duals of linear programming the property that they "price out" the utilized activities so as to leave them profitless, while assigning losses to activities rejected in the optimal solution. In doing so, they assign "subsidies" to some activities.

They do not measure the effects of discrete changes in the requirements vector, i.e. in terms of the right hand side, and in general, do not satisfy the linear programming duality relation (equality for the values of the primal and dual solution).

The total value of the final output goods is not equal to the total imputed value of all original capacities. This is, in fact, one of the crucial differences between integer and linear programming: in integer problems, not all inputs receiving positive prices are completely used up in the optimal solution. As a result, for any problem in which recomputation does not alter the final goods' prices, the Baumol and Gomory prices will impute to the original capacities a value in excess of the total value of the outputs of the industry in the optimal integer solution.

Hence, these recomputed dual prices function only partially as shadow prices. They do not assign a zero value to constraints whose complete omission would not alter the integer optimum. A zero value may be assigned to a constraint which is "absolutely redundant," i.e. one whose omission from any subset of the original constraints would not alter the ability of the subset to produce the integer optimum. Alternatively, a zero value may be assigned to a constraint which is

redundant only given the other constraints, and hence, it may receive a positive dual value in a solution utilizing a different subset of constraints.

Whenever a subset of original constraints is sufficient to generate the required cutting planes, and hence yields the same integer optimum as would the complete set of restrictions, then the computation corresponds to a process of backward pivoting, and the duals so obtained are independent of the choice of the cutting planes. However, even in this situation it may be that several subsets of n constraints are sufficient in which case more than one set of "unique" duals results.

Alcaly and Klevorick (1) attempted to alter and extend the Gomory and Baumol method of recomputation of the dual prices for integer programs to make these prices more economically satisfying. As an example, they consider the problem facing economy-level planners who are determining the outputs of an industry that consists of n firms, each operating under identical technological conditions. They consider the industry problem in anticipation of the nature of the subsidies and penalties that will emerge in the solution of the problem.

One of the main problems of the Baumol and Gomory dual prices is the fact, that they do not accurately represent the marginal revenue products of the inputs and that they often given zero prices to goods which the economist does usually not consider free goods.

Alcaly and Klevorick's system of generalized dual prices under certain conditions provides a set of prices more consistent with the economical concept of a free good, which we discussed earlier. They

note, however, that in the case of integer programming this definition of a free good is itself open to criticism. They finally mention the possibility that in the case of decentralized planning under conditions of an integer programming nature, the notion of a single set of prices, to which the economist has become so attached, may have to be abandoned.

An Example Problem

To illustrate the concepts developed in this chapter, we will present now a numerical example, involving two discrete and one continuous variable. The primal problem under consideration is a maximization problem, subject to three inequality constraints, i.e. we deal with a mixed-integer programming problem.

(A) Mixed-Integer Program

$$\begin{array}{ll}
 \text{max } z = 2x_1 + 2x_2 + 5x_3 & \\
 \text{(MIP)} \quad \text{subject to} & \\
 3x_1 + 2x_2 + x_3 \leq 6 & \\
 -x_1 + x_2 + 4x_3 \leq 8 & \\
 x_1 + x_2 + 2x_3 \leq 6 & \\
 x_1, x_2 \quad \text{integer, } \geq 0 & \\
 x_3 \geq 0 & .
 \end{array}$$

By one of the existing methods, here also by exhaustive enumeration, the optimal solution of this problem can be determined as:

$$\begin{aligned}\bar{x}_1 &= 1 & \bar{x}_3 &= 9/4 \\ \bar{x}_2 &= 0 & \bar{z} &= 13 \frac{1}{4} = 53/4 \quad .\end{aligned}$$

(B) Linear Program

After relaxing the integrality constraints, we obtain the following problem

$$\begin{aligned} \text{(LP)} \quad & \max 2x_1 + 2x_2 + 5x_3 \\ & \text{subject to} \\ & 3x_1 + 2x_2 + x_3 \leq 6 \\ & -x_1 + x_2 + 4x_3 \leq 8 \\ & x_1 - x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \quad . \end{aligned}$$

Three iterations, using the Simplex Method, yield the following optimal solution:

$$\begin{aligned} z^* &= 14 & x_1^* &= \frac{16}{13} \\ & & x_2^* &= 0 \\ & & x_3^* &= \frac{30}{13} \quad . \end{aligned}$$

This solution is obviously different from the solution to (A).

(C) Equivalent Linear Program

According to the rules developed in earlier parts of this chapter, the equivalent linear program to (MIP) has the form

$$\begin{aligned} & \max (2 + s_1)x_1 + (2 + s_2)x_2 + 5x_3 \\ \text{(ELP)} \quad & \text{subject to} \end{aligned}$$

$$3x_1 + 2x_2 + x_3 \leq 6$$

$$-x_1 + x_2 + 4x_3 \leq 8$$

$$x_1 + x_2 + 2x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0.$$

To determine the subsidies (penalties) s_1, s_2 we have to solve the dual of (MIP), (DIP), which reads:

$$\begin{aligned} & \max \quad \min \quad 6m_1 + 8m_2 + 6m_3 - n_1x_1 - n_2x_2 \\ & \quad \quad \quad x_1, x_2 \quad m_1, m_2, m_3 \\ & \quad \quad \quad \text{subject to} \end{aligned}$$

$$\text{(DIP)} \quad 3m_1 - m_2 + m_3 - n_1 = 2$$

$$2m_1 + m_2 - m_3 - n_2 = 2$$

$$m_1 + 4m_2 + 2m_3 - n_3 = 5$$

$$n_3, m_1, m_2, m_3 \geq 0$$

$$x_1, x_2 \quad \text{integer}$$

$$n_1, n_2 \quad \text{unrestricted}$$

This max-min optimization problem becomes a regular linear programming problem, if we use the optimal solution $\bar{x}_1, \bar{x}_2, \bar{x}_3$ to (MIP). Incorporating the first two constraints of the dual (which correspond to the integer primal variables x_1, x_2) into the objective function, we get:

$$\begin{aligned} \min_{m_1, m_2, m_3} \quad & 6m_1 + 8m_2 + 6m_3 - (3m_1 - m_2 + m_3 - 2)\bar{x}_1 \\ & - (2m_1 + m_2 + m_3 - 2)\bar{x}_2 \end{aligned}$$

subject to

$$m_1 + 4m_2 + 2m_3 - n_3 = 5$$

$$m_1, m_2, m_3 \geq 0.$$

Using $\bar{x}_1 = 1$, $\bar{x}_2 = 0$, the problem becomes

$$\begin{aligned} \min_{m_1, m_2, m_3} \quad & 6m_1 + 8m_2 + 6m_3 - (3m_1 - m_2 + m_3 - 2) \\ & = 3m_1 + 9m_2 + 5m_3 + 2 = w \end{aligned}$$

subject to

$$m_1 + 4m_2 + 2m_3 \geq 5$$

$$m_1, m_2, m_3 \geq 0.$$

This problem has the optimal solution

$$\begin{aligned} \bar{m}_1 &= 0 & \bar{m}_3 &= 0 \\ \bar{m}_2 &= 5/4 & \bar{w} &= 53/4. \end{aligned}$$

We use this result to evaluate the dual surplus variables in (DIP):

$$\bar{n}_1 = 3\bar{m}_1 - \bar{m}_2 + \bar{m}_3 - 2 = -13/4$$

$$\bar{n}_2 = 2\bar{m}_1 + \bar{m}_2 - \bar{m}_3 - 2 = -3/4$$

$$\bar{n}_3 = \bar{m}_1 + 4\bar{m}_2 + 2\bar{m}_3 - 5 = 0.$$

Applying the rules stated in the preceding chapter, we now are able to evaluate the subsidies (penalties) in (ELP):

$$\begin{aligned}x_1 &= 1 & s_1 &= \bar{n}_1 = -13/4 \\x_2 &= 0 & s_2 &= \min(6, \bar{n}_2) = -3/4,\end{aligned}$$

i.e. the activities 1 and 2 are both penalized, whereas activity 3 remains unaffected, since it corresponds to a continuous variable. Inserting these values into (ELP), it becomes

$$\max (2 - 13/4) x_1 + (2 - 3/4)x_2 + 5x_3 = 5/4x_1 + 5/4x_2 + 5x_3 = z_L$$

subject to

$$\begin{aligned}3x_1 + 2x_2 + x_3 &\leq 6 \\-x_1 + x_2 + 4x_3 &\leq 8 \\x_1 - x_2 + 2x_3 &\leq 6, \quad x_1, x_2, x_3 \geq 0.\end{aligned}$$

This problem has alternate optimal solutions. One of these optimal solutions is:

$$\begin{aligned}\bar{x}_1 &= 1 & \bar{x}_3 &= 9/4 \\ \bar{x}_2 &= 0 & \bar{z}_L &= 10,\end{aligned}$$

i.e. the optimal solution to (MIP) is also optimal to (ELP). The difference in the value of the objective function is caused by the different objective function coefficients.

CHAPTER III

WEINGARTNER'S BASIC CAPITAL BUDGETING MODEL

3.1 Introduction

The analysis for investment decisions by firms has become more sophisticated through the application of new mathematical tools. In particular, the use of mathematical programming permits the whole set of investment alternatives to be considered as a program. Complex interrelationships among investment projects can be stated and analyzed at one time as can the financial relationships imposed by capital rationing. Models which optimize project selection within an investment program allow simultaneous and consistent evaluation of alternatives, even when projects are not independent, and despite capital expenditure and other resource limitations. Among the problems arising in the formulation of such models is the question of which discount rate to use for calculating present values or future values of the cash flows considered. It has been shown that in the presence of certain market imperfections, the use of the company's cost of capital may not lead to an optimization of its economic objectives (15).

In this chapter, we will deal with a capital budgeting model that avoids the mentioned problem of determining or defining the cost of capital, simply by assuming that the appropriate discount rates are available. First, the linear programming formulation, due to Weingartner (15), will be presented, and then Balas's duality concept will be used to cope with the additional assumption of indivisible investment

projects. This assumption leads to a formulation of the model in the form of a pure integer program. Using the results of the preceding chapter, an economic interpretation of the optimal solution will be given that differs significantly from that of the original model.

3.2 The Linear Programming Formulation

The situation which Weingartner considers in his Basic Model (15) involves the allocation of limited amounts of capital among a specified set of n investment opportunities, with the goal of selecting those projects whose total present value is a maximum, but whose total outlay in each period falls within the budget limitations. A planning horizon, T , divided into a finite number of periods, is considered and it is assumed that, within a certain period, the exogenous and internal condition remain constant. This implies that some situations may require a large number of periods in order to provide a realistic representation of the real world. This goal of obtaining a good approximation of existing conditions, however, will have to be weighed against the increased size of the problem and hence the probably increased difficulty of solution that a larger number of periods would cause.

The model considered is deterministic, i.e. all information about cash flows and budget limitations up to the planning horizon is assumed to be known with certainty. As already mentioned above, it is also (implicitly) assumed that the necessary discount rates are given.

Letting

c_{tj} be the costs of project j in time period t ,

C_t be the budget ceiling in time period t ,

b_j be the present value of all cash flows (revenues and costs) associated with project j , and

x_j be the fraction of project j to be undertaken (x_j is the decision variable),

the mathematical statement of the problem is

$$\max \sum_{j=1}^n b_j x_j$$

(LP) subject to

$$\sum_{j=1}^n c_{tj} x_j \leq C_t, \quad t = 1, \dots, T$$

$$0 \leq x_j \leq 1, \quad j = 1, \dots, n.$$

The upper limits of unity on each x_j exclude the possibility of multiple projects, the omission of such a limitation would lead to allocating the entire budget to multiples of the most desirable projects. The model looks implicitly at all combinations of projects, not just one project at a time, to select that set whose total present value is a maximum; it does not eliminate fractional projects from the solution since that would require an additional binary restriction on the x_j . Weingartner (15) has shown that an optimum, if it exists, can always be achieved with at most T fractional projects.

3.3 The Dual Linear Program

As discussed in the preceding chapter, the dual program provides a valuable economic interpretation. In our case, (LP) has a dual of the form

$$\begin{aligned}
 & \min \sum_{t=1}^T \rho_t C_t + \sum_{j=1}^n \mu_j \\
 (LD) \quad & \text{subject to} \\
 & \sum_{t=1}^T \rho_t c_{tj} + \mu_j = b_j, \quad j = 1, \dots, n \\
 & \rho_t, \mu_j \geq 0, \quad t = 1, \dots, T.
 \end{aligned}$$

$\bar{\rho}_t$, the optimal value of ρ_t , represents the present value of an additional dollar added to the budget in period t , assuming optimal use of the budget. It is a "marginal" value, in the sense that it is an increment and it depends on the optimal use of the indicated increment. If the ρ_t , the shadow or opportunity costs of the budget constraints, are nonzero, it follows (by complementary slackness) that the budgets are critical (binding). However, these optimal shadow costs will differ from period to period depending on the ability of the firm to utilize an additional dollar of the (currently) submarginal investments in each period.

Each variable ρ_t refers to a budgetary limitation C_t , whereas the remaining variables μ_j are associated with the n constraints $x_j \leq 1$, and will be shown to play a role of budget evaluators. At optimality, we obtain the relation

$$(3.1) \quad \bar{\mu}_j = b_j - \sum_{t=1}^T \bar{\rho}_t c_{tj}.$$

When project j is acceptable, so that $\bar{x}_j > 0$, the corresponding dual constraint is met exactly, and inequality (3.1) becomes an equality which can be read as: the "value" of an accepted project j is the amount by which its present value b_j exceeds the sum of the discounted outlays as

evaluated at the corresponding dual shadow prices of each budget year, $\sum_{t=1}^T \bar{\rho}_t^c t_j$.

If the given linear program is solved by the Simplex Method, and if we denote the evaluators $z_j - b_j$ by σ_j , then

$$(3.2) \quad \bar{z}_j = \sum_{t=1}^T \bar{\rho}_t^c t_j + \bar{\mu}_j$$

and

$$(3.3) \quad \bar{\sigma}_j = \sum_{t=1}^T \bar{\rho}_t^c t_j + \bar{\mu}_j - b_j.$$

With each x_j , there is an associated slack variable t_j , defined by the relation

$$(3.4) \quad t_j \equiv 1 - x_j.$$

If $\bar{t}_j > 0$, then $\bar{\mu}_j = 0$, so that (3.3) may be written as

$$(3.5) \quad \bar{\sigma}_j = \sum_{t=1}^T \bar{\rho}_t^c t_j - b_j \cong 0.$$

If also $0 < \bar{x}_j < 1$ holds, i.e. j is a proper fractional project, the relation (3.3) becomes

$$(3.6) \quad \bar{\sigma}_j = \sum_{t=1}^T \bar{\rho}_t^c t_j - b_j = 0;$$

this means that in such a case, an accepted project must fulfill the condition that its present value, b_j , equals the present value of all future expenditures, $\sum_{t=1}^T \bar{\rho}_t^c t_j$.

Weingartner (15) calls such (proper) fractional projects, which necessarily imply relation (3.6), "marginal for acceptance." In the event that (3.6) is met, but also $\bar{x}_j = 0$, he calls the project "marginally rejected."

If, finally,

$$(3.7) \quad \bar{\sigma}_j = \sum_{t=1}^T \bar{p}_t c_{tj} - b_j > 0,$$

the project j just necessarily be totally rejected, so that $\bar{x}_j = 0$. Such projects show a net present value of benefits which is less than the present value of the expenditures that must be undertaken.

Summarizing, we can say that the optimal values of the dual variables, $\bar{\mu}_j$ and $\bar{\sigma}_j$, appear to be tools for ranking all projects considered. Since the linear programming solution takes into account not only the goodwill of the individual projects, but also their interrelationships through the budget constraints, the ranking order, thus established, may well differ from ranking by present value or rate of return methods. Some of the difficulties arising in the use and interpretation of the linear programming solution are:

There are projects that are essentially discrete, as discussed in Chapter II, and thus fractional solutions are meaningless. Furthermore, the maximum possible number of fractional projects increases as explicit interrelationships between the projects are taken into account. These aspects point into the direction of an integer programming formulation. In order to cope with project indivisibilities, we add binary constraints for the x_j and use Balas's duality concept for a closed

formulation of primal and dual problems as well as for the economic interpretation of their optimal solutions.

3.4 Integer Programming Formulation of Primal and Dual according to Balas's Duality Concept

As indicated in the preceding section, integer programming may be applied to Weingartner's Basic Model (LP) to produce the optimal set of integral investment projects under budget constraints for the given time horizon. The model then appears in the form

$$\begin{aligned}
 & \max \sum_{j=1}^n b_j x_j \\
 (P) \quad & \text{subject to} \\
 & \sum_{j=1}^n c_{tj} x_j \leq C_t, \quad t = 1, \dots, T \\
 & x_j = 0 \text{ or } 1, \quad j = 1, \dots, n
 \end{aligned}$$

This is a pure 0-1 integer programming problem, which may also be written in the form of a regular integer programming problem, including the upper bound constraints on the x_j . In Balas's qualification of the special cases, (P) belongs to type (b), i.e. $M_1 \equiv \emptyset$, $N - N_1 \equiv \emptyset$. The variables x_j , which represent the decision of executing or not executing project j , take only values of 0 and 1, i.e. projects are either fully accepted or fully rejected.

Based on the concepts discussed in Chapter II, we are able to formulate a dual problem (D) of the form

$$\begin{aligned}
 & \max \min \sum_{t=1}^T \rho_t c_t - \sum_{j=1}^n \mu_j x_j \\
 (D) \quad & \text{subject to} \\
 & \sum_{t=1}^T c_{tj} \rho_t - \mu_j = b_j, \quad j = 1, \dots, n \\
 & \rho_t \geq 0, \quad t = 1, \dots, T \\
 & x_j = 0 \text{ or } 1 \quad j = 1, \dots, n \\
 & \mu_j \text{ unrestricted in sign.}
 \end{aligned}$$

(D) is a mixed-integer 0-1 optimization problem of the type max-min over the nonnegative orthant, but is otherwise unconstrained since μ_j is unrestricted in sign. The dual variables ρ_t and the dual surplus variables μ_j are continuous, the μ_j being unrestricted in sign since they correspond to the integer constrained primal variables x_j ("partial relaxation of the constraints").

Using the notational correspondence

Weingartner	Balas
(x_1, \dots, x_n)	x^1
(ρ_1, \dots, ρ_T)	u^2
(c_1, \dots, c_T)	b^2
(b_1, \dots, b_n)	c^1
$\begin{pmatrix} c_{11} & \cdot & \cdot & \cdot & \cdot & c_{1n} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ c_{T1} & \cdot & \cdot & \cdot & \cdot & c_{Tn} \end{pmatrix}$	A^{21}

(P) and (D) read in Balas's notation

$$\begin{aligned}
 & \max c^1 x^1 \\
 (P') \quad & \text{subject to} \\
 & A^{21} x^1 \leq b^2 \\
 & x_j = 0 \text{ or } 1, \quad j \in N_1, \text{ and} \\
 & \max \min u^2 b^2 - v^1 x^1 \\
 & x^1 \quad u^2 \\
 (D') \quad & \text{subject to} \\
 & u^2 A^{21} - v^1 = c^1 \\
 & x_j = 0 \text{ or } 1, \quad j \in N_1 \\
 & v_j \text{ unrestricted, } j \in N_1.
 \end{aligned}$$

Since $A^{12}, A^{22} \equiv \emptyset$, the constraint qualification is met. To avoid the difficulties that arise in establishing a "pricing system" for the problem (P) and (D), we will use the approach presented in Chapter II, i.e. where a system of (linear programming) nonnegative shadow prices u_i is combined with unconstrained quantities s_j , subsidies or penalties, depending on their sign. In a case like the one under consideration, where the primal is a pure 0-1 integer programming problem, we are able to derive some additional properties using the special structure of the model.

Let us consider (P') and (D') in Balas's notation and let (\bar{x}^1) , and (\bar{x}^1, u^{-2}) be a pair of optimal solutions to (P') and (D') respectively. In accordance with Theorem 6, we formulate the equivalent linear program (ELP'), for which any optimal solution to (P'), hence also (\bar{x}^1) , is optimal:

$$\begin{aligned}
 & \max (c^1 + s^1)x^1 \\
 \text{(ELP')} \quad & \text{subject to} \\
 & A^{21}x^1 \leq b^2 \\
 & I x^1 \leq 1 \\
 & x_j \geq 0 \quad j \in N_1,
 \end{aligned}$$

where I is an $n_1 \times n_1$ identity matrix. The dual of (ELP') has the form

$$\begin{aligned}
 & \min (u^2 b^2 + w \cdot 1) \\
 \text{(ELD')} \quad & \text{subject to} \\
 & u^2 A^{21} + w \cdot I \leq c^1 + s^1 \\
 & u^2, w \geq 0.
 \end{aligned}$$

3.5 Properties of Primal and Dual Problems and Economic Interpretation of the Optimal Solutions

As opposed to the linear programming formulation, discussed in the first part of this chapter, there is no immediate interpretation of the dual variables in terms of shadow prices for problem (D). As pointed out in the preceding chapter, however, an "equivalent linear program" can be constructed for which the optimal solution to (P) is also optimal. This linear program and its dual then provide a means of interpreting the primal and dual variables, at optimality, of (P) and (D) in terms of "generalized shadow prices" and "subsidies" and "penalties". Before pursuing this route, we first state some interesting properties of (P) and (D) which follow from Balas's theorems 1 to 5 directly.

Theorem 1 (Involution) holds in the same form as in the general case.

Since the vector of primal slack variables, (y_1, \dots, y_T) , is componentwise separable with respect to the primal decision vector (x_1, \dots, x_n) , Theorem 2 (Saddle-Point) holds. Therefore, if (P) has an optimal solution $(\bar{x}_1, \dots, \bar{x}_n)$, there exists a set of dual variables $(\bar{\rho}_1, \dots, \bar{\rho}_T)$ such that $(\bar{\rho}_1, \dots, \bar{\rho}_T; \bar{x}_1, \dots, \bar{x}_n)$ is an optimal solution to (D) with

$$(3.8) \quad \max \sum_{j=1}^n b_j x_j = \max_{x_j} \min_{\rho_t} \sum_{t=1}^T \rho_t C_t - \sum_{j=1}^n \mu_j x_j$$

or

$$(3.8') \quad \max \sum_{j=1}^n b_j x_j = \max_{x_j} \min_{\rho_t} \sum_{t=1}^T \rho_t C_t - \sum_{j=1}^n \left(\sum_{t=1}^T c_{tj} \rho_t - b_j \right) x_j,$$

i.e. the present value of all accepted projects is, at optimality, equal to the "value" of the resources minus the amount by which the value of all cash outlays exceeds the present value of all accepted projects (both evaluated with the optimal dual variables $\bar{\rho}_t$). Since both the maximization and the max-min optimization processes in (3.8) are to be understood as subject to the relevant constraint sets of primal and dual problems, (3.8) can also be written as

$$(3.8'') \quad \sum_{j=1}^n b_j \bar{x}_j = \sum_{t=1}^T \bar{\rho}_t C_t - \sum_{j=1}^n \left(\sum_{t=1}^T c_{tj} \bar{\rho}_t - b_j \right) \bar{x}_j,$$

which reduces to

$$(3.9) \quad \sum_{t=1}^T \bar{\rho}_t C_t = \sum_{j=1}^n \sum_{t=1}^T c_{tj} \bar{\rho}_t.$$

Relation (3.9) relates to the (optimal) dual variables with the corresponding budgets and cash flows.

Furthermore, the function $F(x_j, \rho_t)$, defined by

$$\begin{aligned}
 (3.10) \quad F(x_j, \rho_t) &\equiv \sum_{j=1}^n b_j x_j + \sum_{t=1}^T \rho_t C_t - \sum_{t=1}^T \rho_t \sum_{j=1}^n c_{tj} x_j \\
 &= \sum_{j=1}^n b_j x_j + \sum_{t=1}^T \rho_t y_t \\
 &= \sum_{t=1}^T \rho_t C_t - \sum_{j=1}^n \mu_j x_j,
 \end{aligned}$$

has a saddle-point at $(\bar{x}_1, \dots, \bar{x}_n; \bar{\rho}_1, \dots, \bar{\rho}_T)$:

$$(3.11) \quad F(x_j, \bar{\rho}_t) \leq F(\bar{x}_j, \bar{\rho}_t) \leq F(\bar{x}_j, \rho_t),$$

for all $x_j \in X(\bar{y}_t)$. $X(\bar{y}_t)$ is the set of all x_j that satisfy the constraint set $\sum_{j=1}^n c_{tj} x_j = C_t - \bar{y}_t$, all t , with \bar{y}_t still standing for the optimal value.

The saddle-point property (3.11) of the function $F(x_j, \rho_t)$ has a possible interpretation in terms of a two-person zero-sum game, where $F(x_j, \rho_t)$ represents a partly discrete (in the x_j), partly continuous (in the ρ_t) payoff-matrix. Player I has all $x_j \in X(\bar{y}_t)$ as his possible strategies, whereas Player II can choose between all ρ_t values that satisfy the dual constraint set.

If we substitute from the definition of $F(x_j, \rho_t)$ and write (3.11) in the form of two inequalities, we obtain the following relations:

$$(3.12) \quad \sum_{j=1}^n b_j x_j + \sum_{t=1}^T \bar{\rho}_t y_t \leq \sum_{j=1}^n b_j \bar{x}_j$$

and

$$(3.13) \quad \sum_{j=1}^n b_j \bar{x}_j \cong \sum_{j=1}^n b_j \bar{x}_j + \sum_{t=1}^T \bar{\rho}_t \bar{y}_t .$$

Relation (3.12) may be interpreted as follows: A value $\bar{\rho}_t$ has been placed on the unused funds y_t in any time-period t , and Player I wants to act so as to maximize the project returns and the value of these unused funds; the best he can get is equality with the right-hand side of (3.12) when he plays the optimal strategy \bar{x}_j . Then, the value of the unused funds will go to zero, a result that will be obtained below, too, using the complementary slackness conditions.

Player II, in (3.13), faces the following situation: A certain (the optimal) set of investment opportunities has been selected which require certain funds and determine the amounts of the unused funds in any period. He wants to select among all feasible values ρ_t those that minimize the value of the fixed unused funds. Player I and Player II may be identified with two different departments in a firm whose goal is to determine the optimal investment policy and the optimal value of the dual evaluators.

Applying Theorem 4, we find: If $(\bar{x}_1, \dots, \bar{x}_n)$ and $(\bar{\rho}_1, \dots, \bar{\rho}_t; \bar{x}_1, \dots, \bar{x}_n)$ are optimal solutions to (P) and (D) respectively, then

$$(3.14) \quad \sum_{t=1}^T \bar{\rho}_t \bar{y}_t = 0,$$

and, since all terms in the sum are nonnegative,

$$(3.15) \quad \bar{\rho}_t \bar{y}_t = 0, \quad \text{all } t.$$

These complementary slackness conditions differ from those in linear programming in that optimality is a sufficient, but not necessary condition for them to hold; consequently, the solution procedure used has to provide other means of identifying the optimal solution.

If we finally assume that $(\bar{x}_1, \dots, \bar{x}_n)$ is a unique, nondegenerate optimal solution to (P), then also (D) has a unique, nondegenerate optimal solution.

Lemma 1. Any optimal solution to the 0-1 integer programming problem (P') is an extreme point of the constraint set of (ELP').

Proof. Assume that \bar{x}^* , the optimal solution to (P'), is not an extreme point of (ELP'). Then \bar{x}^* can be written as a convex combination of the extreme points of (ELP').

Let the optimal extreme point solutions to (ELP') be $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m$; then

$$\bar{x}^* = \sum_{i=1}^m g_i \bar{x}^i, \quad \text{with} \quad \sum_{i=1}^m g_i = 1, \quad g_i \geq 0.$$

Suppose $x_j^* = 1$, then

$$\sum_{i=1}^m g_i x_j^i = 1,$$

which implies

$$x_j^i = 1, \text{ for all } g_i \neq 0.$$

This can be shown by assuming that for at least one i , $x_j^i < 1$ (with $g_i \neq 0$);

then

$$\sum_{i=1}^m g_i x_j^i < \sum_{i=1}^m g_i \cdot 1 = 1,$$

in contradiction with the result above. Suppose now $\bar{x}_j^- = 0$, then

$$\sum_{i=1}^m g_i x_j^i = 0 \quad ,$$

which implies

$$x_j^i = 0, \text{ for all } g_i \neq 0 \quad ,$$

i.e. for all terms that are part of the convex combination.

It follows therefore that \bar{x}^+ cannot be represented as a convex combination of extreme points \bar{x}^i other than the trivial combination involving \bar{x}^+ itself, since \bar{x}^+ is an extreme point of the set $x_j \leq 1$, $x_j \geq 0$, and cannot be written as a convex combination of the other extreme points in this set. Q.E.D.

For a better illustration of the result in Lemma 1, i.e. that the optimal solution to (P') is an extreme point of C, the convex polyhedron formed by the constraint set of (ELP'), we consider the two possible situations that may occur:

(a) The upper bounds on the x_j are more restrictive than the structural constraints; then the convex hull of the constraint set of (P') is identical with the convex set forming C, i.e. the extreme points of C are 0-1 integer points; or:

(b) Some (since we assume the existence of an optimal solution with $\bar{x} \neq 0$, not all) of the structural constraints are more restrictive than the corresponding upper bounds on the x_j ; then the integral valued optimal solution to (P') is one of those remaining extreme points where the structural constraints are not active.

As discussed in Chapter II, the replacement of c^1 in the objective function of (P') by $(c^1 + s^1)$ causes \bar{x}^1 to become optimal with respect to this new objective function. The objective function is now defined on the convex polyhedron C. We can rewrite the constraint set on which C is defined in equation form, by introducing slack variables y_i , $i = 1, \dots, m_2$, and t_j , $j = 1, \dots, n_1$, as follows

$$\begin{aligned}
 & \max (c^1 + s^1)x^1 \\
 \text{(ELP'')} \quad & \text{subject to} \\
 & A^2 x^1 + y^2 = b^2 \\
 & I x^1 + t^1 = 1 \\
 & x^1, y^2, t^1 \geq 0.
 \end{aligned}$$

The above constraint set contains $m_2 + n_1$ equations, thus we need $m_2 + n_1$ variables to form a basis. Since the optimal solution \bar{x}^1 to (P') is integer valued, the corresponding optimal basis for (ELP'') will contain exactly n_1 variables, x_j or t_j , from the set of constraint equations; the remaining m_2 variables in the basis will consequently be the y_i . Hence, though the optimal solution of (ELP'') need generally not be integer, the one which is feasible to (P') can be found by introducing all y_i into the basis. Note that this result holds only for the special case of pure 0-1 integer programming problems (P').

After this discussion of some additional properties of (P'), we turn to our specific problem under consideration, and write the equivalent linear program in Weingartner's notation as

$$\max \sum_{j=1}^n (b_j + s_j) x_j$$

(ELP) subject to

$$\sum_{j=1}^n c_{tj} x_j \leq C_t, \quad t = 1, \dots, T$$

$$x_j \leq 1, \quad j = 1, \dots, n$$

$$x_j \geq 0, \quad j = 1, \dots, n.$$

(ELP) has dual of the form

$$\min \sum_{t=1}^T \rho_t C_t + \sum_{j=1}^n w_j$$

(ELD) subject to

$$\sum_{t=1}^T \rho_t c_{tj} + w_j \leq b_j + s_j, \quad j = 1, \dots, n$$

$$\rho_t, w_j \geq 0, \quad \text{all } j \text{ and all } t.$$

Denoting the surplus variables in (D) by μ_j , and applying the formation rules for the subsidies and penalties s_j , we obtain

$$s_j = \mu_j, \text{ if } \bar{x}_j > 0, \text{ and}$$

$$s_j = \min(0, \bar{\mu}_j), \text{ if } \bar{x}_j = 0,$$

where $\bar{\mu}_j = \sum_{t=1}^T c_{tj} \bar{\rho}_t - b_j \leq 0$ is the value of μ_j in the optimal solution to (D). Then, from Balas's Theorem 6, we obtain: If (P) has an optimal, 0-1 valued integer solution $(\bar{x}_1, \dots, \bar{x}_n)$, there exists a vector of dual shadow prices ρ_t , dual project evaluators w_j , and

subsidies or penalties s_j , with the following properties:

(a) Any optimal solution $(\bar{x}_1, \dots, \bar{x}_n)$ to (P) is an optimal solution to (ELP).

(b) $\sum_{t=1}^T \rho_t c_{tj} + w_j \geq b_j + s_j$, all j , i.e. the funds absorbed by the utilization of the resources plus the funds assigned to the exclusion of multiple projects (upper bounds of 1 on the x_j) are never less than the "recomputed profits," $b_j + s_j$. In other words, none of the projects yields a positive "profit" in the sense defined above.

(c) $\sum_{j=1}^n (\sum_{t=1}^T \rho_t c_{tj} + w_j - b_j - s_j) x_j = 0$. This condition may be split up into n equations, since all terms in the summation over the project index j are nonnegative:

$$(\sum_{t=1}^T \rho_t c_{tj} + w_j - b_j - s_j) x_j = 0, \quad j = 1, \dots, n,$$

i.e. a project is accepted only if the acceptance does not cause a "loss."

(d) $\sum_{t=1}^T \rho_t (C_t - \sum_{j=1}^n c_{tj} x_j) = 0$, or, split up into single terms,

$$\rho_t (C_t - \sum_{j=1}^n c_{tj} x_j) = 0, \quad t = 1, \dots, T. \quad \text{This linear}$$

programming complementary slackness condition is identical to the complementary slackness condition from Theorem 4 for (P) and (D). In

addition to that, however, the analysis yields here $\sum_{j=1}^n w_j (1 - x_j) = 0$,

or, by the same argument as above, $w_j (1 - x_j) = 0$, $j = 1, \dots, n$, i.e.

if the evaluator w_j is positive the project j must be accepted, whereas

for $w_j = 0$, it may or may not be accepted.

$$(e) \quad \sum_{j=1}^n (b_j + s_j) x_j = \sum_{t=1}^T \rho_t C_t + \sum_{j=1}^n w_j;$$

this duality relation between the primal and dual objective functions shows that the system of dual prices ρ_t and project evaluators w_j together with the appropriate system of subsidies and penalties s_j evaluates the accepted projects in a way which is consistent with the total value of the investment program.

As already pointed out, the dual variables now can be interpreted as regular shadow prices, using the concepts of "marginal costs." It will be shown in a later chapter, for the more general case of a mixed-integer program, that not only the primal optimal solution of (P) is optimal to (ELP), but that also the optimal solution of the dual (D) is optimal to (ELD).

The above results show that

- (i) if $\bar{x}_j > 0$, $s_j = \bar{\mu}_j \neq 0$, and
- (ii) if $\bar{x}_j = 0$, $s_j \leq 0$.

In other words, this reads:

- (i) accepted projects may be penalized, subsidized, or left unchanged;
- (ii) rejected projects may have to bear a penalty.

These penalties or subsidies are a direct consequence of the integrality requirement on x_j . They might cause a different ranking of projects than a linear programming approach to the capital budgeting problem, where indivisibilities are not taken into consideration.

CHAPTER IV

WEINGARTNER'S TERMINAL WEALTH MODEL

4.1 Introduction

The complexity of the capital budgeting problem derives from the fact that any set of actions taken today has consequences at later times, and the opportunities available at later dates are related to decisions being implemented currently. While the only decisions that need to be made today are those that require action today, these decisions cannot ignore the range of opportunities at later times. More specific, the decision to utilize resources for the acquisition of assets which yield flows of revenue, but which cannot be turned back into cash or liquid assets without some cost calls for careful analysis.

Using Weingartner's Basic Model for allocating funds to investment opportunities under fixed spending ceilings involved two substantial difficulties. One was determining the budget limits themselves, and the other one was the choice of a discount rate for the purpose of calculating present values of the projects and their outlays. Therefore, Weingartner (15) constructed the Basic Horizon Model (or Terminal Wealth Model) in which some of the quantities that are inputs for the Basic Model are determined by the model simultaneously with the investment decisions.

As in the case of the Basic Model, we will first summarize the main features of the linear model and then consider the consequences of additional 0-1 restrictions on some of the variables. Several

interesting properties of this model will be derived and an acceptance criterion for the suggested projects will be developed. The discussion will be concluded by developing a procedure that allows a certain degree of decentralized decision making in a multi-department organization.

4.2 The Linear Terminal Wealth Model

The model considers the "value of the firm" as of some future time, T , called horizon time. It maximizes this "terminal wealth" of the firm, subject to a cash balance restriction. Beforehand, T must be selected as the point in time prior to which outlays and revenues of potential investments are stated explicitly, but beyond which the actual flows associated with these investments are collapsed into a single quantity, the horizon values. Financial transactions are introduced into the model by means of lending and borrowing without limit at some stated rates of interest r_{lt} and r_{bt} , $t = 1, \dots, T$. Both lending and borrowing are accomplished by means of renewable one-year contracts, where, by convention, all interest is payable at the end of the year.

Letting

a_{tj} be the net cash flow obtainable from acceptance of project j at time t ;

\hat{a}_j be the time T present value of post T cash flows, if any, from project j ;

M_t be the amount of cash made available from projects outside this analysis and from other outside sources at time t ;

$\ell_t = 1 + r_{\ell t}$, where $r_{\ell t}$ is the lending rate of interest from time t to $t + 1$;

$b_t = 1 + r_{bt}$, where r_{bt} is the borrowing rate of interest from time t to $t + 1$;

x_j be the decision variable for the fraction of project j adopted;

w_t be the decision variable for the cash to be borrowed from time t to $t + 1$;

v_t be the decision variable for the cash to be lent from time t to $t + 1$,

the mathematical statement of the problem is

$$\max \sum_{j=1}^n a_j x_j + v_T - w_T$$

(LP) subject to

$$-\sum_{j=1}^n a_{tj} x_j - \ell_{t-1} v_{t-1} + v_t +$$

$$b_{t-1} w_{t-1} - w_t \leq M_t, \quad t = 1, \dots, T$$

$$v_t, w_t \geq 0, \quad t = 1, \dots, T$$

$$0 \leq x_j \leq 1, \quad j = 1, \dots, n.$$

The T cash balance restrictions in this model are to be understood as follows: the net cash outflow to projects minus the cash inflow from time $t - 1$ loans plus the cash outflow from time t loans plus the cash outflow for time $t - 1$ borrowing minus the cash inflow from

time t borrowing must be less or equal to the cash available from outside sources at time t . Although no upper bounds on the amounts borrowed in each period are included, we do not deal in a perfect market since borrowing and lending rate are (in general) different.

The objective function contains basically two components, the net amount of financial assets accumulated at the horizon, $v_T - w_T$, and the post horizon cash flows, discounted back to the horizon. To the restrictions in (LP) one may add those expressing relationships of complementarity and competitiveness between projects (mutually exclusive, contingent projects, etc.). The model may be extended by including restrictions on the amounts borrowed, either in form of limiting constraints on the w_t or by using a rising supply curve for funds; in the latter case, the higher rates of interest associated with larger amounts borrowed can be interpreted as a form of risk premium. This generalization as well as the addition of a dividend policy will be accomplished by one of the following models in this research.

4.3 The Dual Linear Program

The Terminal Wealth Model (LP) has a dual of the form

$$\min \sum_{t=1}^T \rho_t M_t + \sum_{j=1}^n \mu_j$$

(LD) subject to

$$-\sum_{t=1}^T a_{tj} \rho_t + \mu_j \cong \hat{a}_j, \quad j = 1, \dots, n$$

$$\rho_t - \ell_t \rho_{t+1} \cong 0, \quad t = 1, \dots, T-1$$

$$-\rho_t + b_t \rho_{t+1} \leq 0, \quad t = 1, \dots, T-1$$

$$\rho_T \leq 1$$

$$-\rho_T \leq -1$$

$$\rho_t, \mu_j \leq 0, \quad \text{all } t \text{ and all } j.$$

Rewriting the second and third set of constraints and combining them yields

$$\lambda_t \leq \rho_t / \rho_{t+1} \leq b_t.$$

Similar to the case of the Basic Model, one may recognize the meaning of the ρ_t as evaluators of the budget constraints. If budgets are critical, then, from the complementary slackness conditions, we know that these optimal $\bar{\rho}_t$ values may be nonzero. However, these values will differ (in general) from period to period, depending on the ability of the firm to use an additional dollar on the remaining investments in each period.

4.4 Mixed-Integer Programming Formulation

By adding the assumption that the investment opportunities are indivisible in nature, i.e. by restricting the x_j to take only the values 0 or 1, we arrive at the following statement of the problem:

$$\max \sum_{j=1}^n \hat{a}_j x_j + v_T - w_T \quad (4.1)$$

(P) subject to

$$\begin{aligned}
& - \sum_{j=1}^n a_{tj} x_j - \ell_{t-1} v_{t-1} + v_t + \\
& b_{t-1} w_{t-1} - w_t + y_t = M_t, \quad \text{all } t \\
& v_t, w_t, y_t \geq 0, \quad \text{all } t \\
& x_j = 0 \text{ or } 1, \text{ all } j \\
& v_0, w_0 \geq 0.
\end{aligned} \tag{4.2}$$

This is a mixed 0-1 integer programming problem, i.e. in Balas's qualification of special cases it may be assigned to group (a) with $M_1 \equiv \emptyset$. The solution to the problem results in the selection of the optimal set of indivisible investments, i.e. those that maximize the present value of the firm, and the optimal lending and borrowing amounts for all periods up to the horizon time. The variables y_t are the primal slack variables which, under certain assumptions, will be shown to vanish in an optimal solution.

According to Balas's duality concept, the dual to (P) can be written in the form

$$\max_{x_j} \min_{\rho_t} \sum_{t=1}^T M_t \rho_t - \sum_{j=1}^n \mu_j x_j \tag{4.3}$$

(D) subject to

$$- \sum_{t=1}^T a_{tj} \rho_t - \mu_j = a_j, \quad j = 1, \dots, n \tag{4.4}$$

$$\rho_t - \ell_t \rho_{t+1} \geq 0, \quad t = 1, \dots, T-1 \tag{4.5}$$

$$-\rho_t + b_{t \ t+1} \cong 0, \quad t = 1, \dots, T-1 \quad (4.6)$$

$$\rho_T = 1 \quad (4.7)$$

$$-\rho_T = -1 \quad (4.8)$$

$$\rho_t \cong 0, \text{ all } t$$

$$\mu_j \text{ unrestricted, } x_j = 0 \text{ or } 1, \text{ all } j$$

(P) and (D) read in Balas's notation

$$\max c^1 x^1 + c^2 x^2$$

(P) subject to

$$A^{21} x^1 + A^{22} x^2 \leq b^2$$

$$x_j = 0 \text{ or } 1, \quad j \in N_1$$

$$x_j \cong 0, \quad j \in N - N_1$$

and

$$\max_{x^1} \min_{u^2} u^2 b^2 - v^1 x^1$$

subject to

$$u^2 A^{21} - v^1 = c^1$$

$$u^2 A^{22} - v^2 = c^2$$

$$x_j = 0 \text{ or } 1, \quad j \in N_1$$

$$v_j \text{ unrestricted, } j \in N_1$$

$$u_1, v_j \geq 0, \text{ i.e. } M - M_1, j \in N - N_1.$$

4.5 Analysis of Primal and Dual Problems

Using the results of Chapter II, we can immediately state the following properties:

According to Theorem I (Involution) the dual of the dual is the primal. As in the case of the Basic Model, the vector of primal slack variables, (y_1, \dots, y_T) , is componentwise separable with respect to the vector of primal discrete variables, (x_1, \dots, x_n) , and therefore Theorem II holds:

If (P) has an optimal solution $(\bar{x}_1, \dots, \bar{x}_n; \bar{v}_1, \dots, \bar{v}_T; \bar{w}_1, \dots, \bar{w}_T)$, there exists a set of dual prices $(\bar{\rho}_1, \dots, \bar{\rho}_T)$ such that $(\bar{\rho}_1, \dots, \bar{\rho}_T; \bar{x}_1, \dots, \bar{x}_n)$ is an optimal solution to (D), with

$$\max \sum_{j=1}^n \hat{a}_j \bar{x}_j + \bar{v}_T - \bar{w}_T = \max_{x_j} \min_{\rho_t} \sum_{t=1}^T M_t \bar{\rho}_t - \sum_j \bar{\mu}_j \bar{x}_j,$$

i.e., the maximum terminal wealth is equal to the optimal value of the resources (budgets) minus the losses (or profits)¹ associated with the accepted projects.

In addition to that, the function

$$\begin{aligned} F(x_j, v_t, w_t; \rho_t) = & \sum_{j=1}^n \hat{a}_j x_j + v_T - w_T + \sum_{t=1}^T M_t \rho_t + \\ & + \sum_{t=1}^T \rho_t \sum_{j=1}^n a_{tj} x_j \end{aligned}$$

has a saddle-point at $(\bar{x}_1, \dots, \bar{x}_n; \bar{v}_1, \dots, \bar{v}_T; \bar{w}_1, \dots, \bar{w}_T; \bar{\rho}_1, \dots, \bar{\rho}_T)$ of the

¹ losses and profits in the sense introduced in Section 2.2.D

form

$$F(x_j, v_t, w_t; \bar{\rho}_t) \cong F(\bar{x}_j, \bar{v}_t, \bar{w}_t; \bar{\rho}_t) \cong F(\bar{x}_j, \bar{v}_t, \bar{w}_t; \rho_t),$$

or, explicitly,

$$\begin{aligned} & \sum_{j=1}^n \hat{a}_j x_j + v_T - w_T + \sum_{t=1}^T \bar{\rho}_t y_t \\ & \cong \sum_{j=1}^n \hat{a}_j \bar{x}_j + \bar{v}_T - \bar{w}_T \\ & \cong \sum_{t=1}^T \rho_t M_t - \sum_{j=1}^n \mu_j \bar{x}_j - \sum_{t=1}^{T-1} \bar{v}_t (\rho_t - \ell_t \rho_{t+1}) \\ & \quad - \sum_{t=1}^{T-1} \bar{w}_t (-\rho_t + b_t \rho_{t+1}) - (\rho_T - 1) \bar{v}_T + (\rho_T - 1) \bar{w}_T, \end{aligned}$$

for all $(x_j, v_t, w_t) \in X(\bar{y}_t)$, and all $\rho_t \in U(\bar{v}_t)$. Here $X(\bar{y}_t)$ is the set of all primal variables satisfying the primal constraint set, when the primal slack vector is fixed at its optimal value $(\bar{y}_1, \dots, \bar{y}_T)$, and $U(\bar{v}_t)$ is the set of all dual variables ρ_t satisfying the dual constraints (4.5), (4.6), (4.7), and (4.8), when the dual surplus vector (that corresponds to continuous primal variables) is fixed at its optimal value $(\bar{v}_1, \dots, \bar{v}_T)$.

The above two inequalities may be read: the maximum terminal wealth of the firm is not less than the terminal wealth using any investment policy, plus the value of the excess capacities, evaluated at the optimal dual prices; and not more than the value of the resources using any feasible dual prices, minus the losses that occur if the optimal investment and lending-borrowing policies are combined with such nonoptimal dual prices. Again, as for the Basic Model, a game-theoretical

interpretation may be given, with $\sum_{j=1}^n \hat{a}_j \bar{x}_j + \bar{v}_T - \bar{w}_T$ being the value of the game.

According to Balas's Theorem 4, the following complementary slackness conditions hold

$$\bar{\rho}_t \bar{y}_t = 0, \quad t = 1, \dots, T \quad (4.9)$$

$$(\bar{\rho}_t - \ell_t \bar{\rho}_{t+1}) \bar{v}_t = 0, \quad t = 1, \dots, T - 1 \quad (4.10)$$

$$(-\bar{\rho}_t + b_t \bar{\rho}_{t+1}) \bar{w}_t = 0, \quad t = 1, \dots, T - 1 \quad (4.11)$$

$$\bar{v}_T - \bar{w}_T - \sum_{t=1}^T M_t \bar{\rho}_t - \sum_{t=1}^T \bar{\rho}_t \sum_{j=1}^n a_{tj} \bar{x}_j = 0. \quad (4.12)$$

Equations (4.9), (4.10), (4.11) will be discussed in more detail in later parts of this chapter. The complementary slackness condition (4.12) can be understood as follows: the net cash outflow (lending minus borrowing) at the horizon time is, in the case of optimality, equal to the value of the resources plus the cash flows associated with indivisible projects only.

Using Theorem 5, we can finally state that the existence of a unique nondegenerate optimal solution to (P) implies the existence of a unique optimal solution to (D).

In the following part of this chapter, some further properties of the model will be derived and stated in form of three lemmas; based on these, an acceptance criterion for the suggested projects will be developed. Lopez (17) obtained some results that will be used in this context.

Lemma 2. Under all explicit and implicit assumptions made (we will use here, in particular, the fact that $\ell_t, b_t > 0$), the following relations hold

- (a) $\rho_t > 0$, all t
- (b) $\rho_T = 1$
- (c) $b_t \geq \ell_t$, $t = 1, \dots, T - 1$, for a feasible solution
- (d) $\bar{y}_t = 0$, all t .

Proof.

- (b) Combining the constraints (4.7), (4.8) we obtain

$$1 \leq \rho_T \leq 1, \text{ and thus } \rho_T = 1.$$

- (a) Using the constraint set (4.5)

$$\rho_t \geq \ell_t \rho_{t+1} \quad \text{we find that } \rho_t \geq \left(\prod_{r=t}^{T-1} \ell_r \right) \rho_T$$

and, with the result (b),

$$\rho_t \geq \prod_{r=t}^{T-1} 1_r; \text{ since } \ell_t > 0, \text{ all } t, \rho_t > 0, \text{ all } t.$$

- (c) Considering the constraint sets (4.5) and (4.6)

$$\ell_t \rho_{t+1} \leq \rho_t \leq b_t \rho_{t+1}, \quad t = 1, \dots, T - 1,$$

$$\ell_t \leq b_t, \quad t = 1, \dots, T - 1 \text{ necessarily follows for all}$$

feasible solutions.

If, on the other hand, $\ell_t > b_t$, for some t , then the constraint set is violated, the dual has no feasible solution, and the primal is either not feasible or unbounded. In economic terms, the solution will then be to borrow money at b_t , in period t , and lend it away at ℓ_t , without limits. The model fails to find this result at the horizon time T .

- (d) Using result (a), the complementary slackness condition

$$\bar{\rho}_t \bar{y}_t = 0, \text{ all } t, \text{ implies } \bar{y}_t = 0, \text{ all } t. \quad \text{Q.e.d. (4.9)}$$

In other words, the primal cash balance restrictions can be considered as equality constraints. In the optimal case, the policy will always be to use all the available funds.

Lemma 3. If $l_t < b_t$, then $\bar{w}_t \bar{v}_t = 0$, for $t = 1, \dots, T - 1$.

Proof.

The proof will be given for the two possible cases:

(a) $\bar{v}_t > 0$. This implies, by means of relation (4.10), that

$\rho_t = l_t \rho_{t+1}$; then $\rho_t \neq b_t \rho_{t+1}$, and, by means of relation (4.11), $\bar{w}_t = 0$. Here, the firm is lending money, from period t to $t + 1$.

(b) $\bar{w}_t > 0$. This implies, by means of relation (4.11), that

$\rho_t = b_t \rho_{t+1}$; then $\rho_t \neq l_t \rho_{t+1}$, and, by means of relation (4.10), $\bar{v}_t = 0$. Here, the firm is borrowing money, from t to $t + 1$. In both cases,

$\bar{w}_t \cdot \bar{v}_t = 0$. Q.e.d.

According to the results above, it will never be optimal to borrow and lend money in the same period, if $l_t < b_t$. The optimal policy will be either only to lend or only to borrow money in period t . If $l_t = b_t$, we have indifference for changes of v_t and w_t to $v_t + \delta$, $w_t + \delta$. This is true because $\rho_t = l_t \rho_{t+1} = b_t \rho_{t+1}$, by (4.11) and (4.10) i.e. these two constraints can be summarized on one equality constraint, and it is possible to change the primal variable $(v_t - w_t)$, that corresponds to this one constraint, by an unrestricted variable. The analogous argument holds for $v_T - w_T$; regardless of the respective values for l_t and b_t .

Before we state the next lemma, let us consider problem (P), if we select a certain vector $(x_j) = (\bar{x}_j)$. We then obtain a linear programming

problem of the form:

$$\max v_T - w_T + \sum_{j=1}^n \hat{a}_j \bar{x}_j$$

(LPR) subject to

$$- \ell_{t-1} v_{t-1} + v_t + B_{t-1} w_{t-1} - w_t + y_t = M_t + \sum_{j=1}^n a_{tj} \bar{x}_j$$

$$v_t, w_t, y_t \geq 0$$

with the dual

$$\min \sum_{t=1}^T \rho_t (M_t + \sum_{j=1}^n a_{tj} \bar{x}_j)$$

(LDU) subject to

$$\rho_T = 1$$

$$\rho_t \geq \ell_t \rho_{t+1}, \quad t = 1, \dots, T-1$$

$$\rho_t \leq b_t \rho_{t+1} \quad t = 1, \dots, T-1$$

$$\rho_t \geq 0, \quad \text{all } t.$$

Lemma 4. The optimal solution to (D) has the property that either

$$\rho_t = \ell_t \rho_{t+1}, \quad t = 1, \dots, T-1$$

or

$$\rho_t = b_t \rho_{t+1}, \quad t = 1, \dots, T-1$$

or both.

Proof.

If we combine equations (4.3) and (4.4), i.e. incorporate the first set of constraints into the objective function, we obtain the dual

problem (D) in the form

$$\max_{x_j} \min_{\rho_t} \sum_{j=1}^n a_j x_j + \sum_{t=1}^T \rho_t (M_t + \sum_{j=1}^n a_{tj} x_j) \quad (4.13)$$

(D1) subject to

$$\rho_t \cong \ell_t \rho_{t+1}, \quad t = 1, \dots, T-1 \quad (4.14)$$

$$\rho_t \cong b_t \rho_{t+1}, \quad t = 1, \dots, T-1 \quad (4.15)$$

$$\rho_T = 1$$

$$x_j = 0 \text{ or } 1, \quad \text{all } j$$

$$\rho_t \cong 0, \quad \text{all } t.$$

If we now select a vector (\bar{x}_j) , then we get the linear programming problem

$$\min_{\rho_t} \sum_{t=1}^T \rho_t (M_t + \sum_{j=1}^n a_{tj} \bar{x}_j)$$

(D2) subject to

$$\rho_T = 1$$

$$\rho_t \cong \ell_t \rho_{t+1}, \quad t = 1, \dots, T-1$$

$$\rho_t \cong b_t \rho_{t+1}, \quad t = 1, \dots, T-1$$

$$\rho_t \cong 0,$$

which is identical with (LDU). In (D2), the constraint set is formed by $(2T - 2)$ inequalities; therefore, we can have at most $(2T - 2)$ variables, including slacks, at a nonzero level.

From Lemma 2, we know that $\rho_2 > 0$, $t = 1, \dots, T-1$, so that $(T - 1)$ other variables are left that can be greater than zero. We now

distinguish between the following two cases:

(a) $b_t \neq \ell_t$. Then at least one of the slack variables of (4.14), (4.15), must be greater than zero. For $(T - 1)$ constraints, this yields $(T - 1)$ nonzero slacks, which completes already the maximum number of $(2T - 2)$ nonzero variables. Consequently, it is impossible for both slack variables of (4.14) and (4.15) to be greater than zero, i.e.

$$(\rho_t - \ell_t \rho_{t+1})(-\rho_t - b_t \rho_{t+1}) = 0.$$

(b) $\ell_t = b_t$. Then the corresponding pair of constraints (4.14), (4.15) degenerates to one single constraint

$$\rho_t = \ell_t \rho_{t+1} \quad \text{or} \quad (\rho_t - \ell_t \rho_{t+1}) = 0. \quad \text{Q.e.d.}$$

We know that, if $b_t = \ell_t$, there exists always a solution to (D2), whichever vector (\bar{x}_j) was selected, since changes of the price vector in the objective function do not affect feasibility.

From Lemma 4, we know that, at least, either the constraint corresponding to the borrowing variable w_t or the one corresponding to the lending variable v_t has to be an equality (for $t = 1, \dots, T - 1$).

Let us therefore define $\bar{\rho}_t$, such that $\bar{\rho}_t = \ell_t \bar{\rho}_{t+1}$, or $\bar{\rho}_t = b_t \bar{\rho}_{t+1}$, and $\bar{\rho}_T = 1$. Then it is possible to write problem (D1), without the original constraint set, as

$$\max_{x_j} \sum_{j=1}^n \hat{a}_j x_j + \min \sum_{t=1}^T \bar{\rho}_t (M_t + \sum_{j=1}^n a_{tj} x_j)$$

or, using a different notation,

$$(D3) \quad \max_{x_j} \left\{ x_0 \equiv \sum_{j=1}^n \hat{a}_j x_j + \min_{\bar{\rho}_t} \sum_{t=1}^T \bar{\rho}_t (M_t + \sum_{j=1}^n a_{tj} x_j) \right\}.$$

Problem (D3) can be handled in an iterative procedure, as we will discuss in a later part of this chapter. The next step will be to develop, for the special case of a perfect market, i.e. when $\ell_t = b_t$, all t , an acceptance criterion that allows an evaluation of the suggested projects without solving the programming problem. Let us denote the common interest rate for period t with i_t (with $i_t + 1 = r_t = \ell_t = b_t$). Then the inequalities (4.14), (4.15) become the equalities

$$\begin{aligned} \rho_t &= r_t \rho_{t+1} \quad \text{or} \\ \rho_t &= \rho_T \prod_{i=t}^{T-1} r_i, \end{aligned}$$

which becomes,

$$\rho_t = \prod_{i=t}^{T-1} r_i. \quad (4.17)$$

Using (4.17), problem (D3) can be written as

$$\max_{x_j} \left\{ x_0 \equiv \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \prod_{i=1}^{T-1} r_i (m_t + \sum_{j=1}^n a_{tj} x_j) \right\},$$

or, after rearranging the terms,

$$(D4) \quad \max_{x_j} \left\{ x_0 \equiv \sum_{t=1}^T \left(\prod_{i=1}^{T-1} r_i \right) M_t + \sum_{j=1}^n x_j \left(\hat{a}_j + \sum_{t=1}^T a_{tj} \left(\prod_{i=t}^{T-1} r_i \right) \right) \right\}.$$

Note that the expression in the second parenthesis, after x_j ,

$$A_j \equiv \hat{a}_j + \sum_{t=1}^T a_{tj} \left(\prod_{i=t}^{T-1} r_i \right)$$

is the present value at $t = T$ of all cash flows associated with project j . The first term in (D4),

$$\sum_{t=1}^T \left(\prod_{i=t}^{T-1} r_i \right) M_t$$

is a constant, independent of x_j , so that the optimal value of x_j (the constraints have been incorporated in the objective function) will depend only on the sign of A_j .

Consequently, the criterion for accepting or rejecting investment projects can be formulated as:

$$\text{if } A_j \equiv a_j + \sum_{t=1}^T a_{tj} \left(\prod_{i=t}^{T-1} r_i \right) \begin{cases} \geq 0, & \text{then } x_j = 1, \text{ accept} \\ < 0, & \text{then } x_j = 0, \text{ reject} \end{cases}$$

project j .

This result is equivalent to the familiar rule of accepting or rejecting a project in accordance with its present value at the horizon ("future value") being ≥ 0 or < 0 , respectively.

4.6 Decentralized Decision Making

In Chapter II we discussed the use of linear programming dual variables for decentralizing the decisions in a multi-division corporation. The central authority need only to compute a set of dual prices which satisfies certain conditions and order division managers to maximize their profits. The result would then be a partially self-policing system for the achievement of an optimal allocation of resources. One would achieve the economic benefits of central direction (in the form of central decisions about the prices of products and the accounting prices for scarce resources) without the costly administrative burden and

unpleasant bureaucratic interference that goes with detailed central supervision.

As pointed out, such an approach encounters serious problems in the case of mixed-integer or pure integer programs. We will show, however, that for the Terminal Wealth Model, a decentralized decision making process can be developed that avoids the involvement of the central authority in the evaluation of divisional investment decision problems.

Before conducting the mathematical analysis, let us first give a brief description of the approach to be used. Corporate headquarters is faced with the problem of determining the optimal set of indivisible investments and the optimal financing policy (lending to and/or borrowing from the capital market) for a multi-division corporation. Every single division (department) has at its disposal the cash flow patterns associated with all investment opportunities (the cash flows from year one up to the horizon time and the horizon time value of discounted post horizon cash flows).

Now headquarters passes out a set of dual prices that are consistent with corporate finance policies and that fulfill certain requirements. Then every division uses this set of accounting prices in a criterion to evaluate its projects, and determine which projects are acceptable and which have to be rejected.

The cash flow pattern only for acceptable projects are presented to headquarters, where this information is pooled and a corporate decision problem is constructed; the solution to this problem yields an "over-all" set of acceptable investment projects. On the basis of these projects,

a new set of dual prices can be determined, which are then passed out again to the single divisions to evaluate their opportunities, etc. The procedure continues until, after a finite number of steps, a set of investment projects maximizing the corporation's profits and a set of internally generated optimal dual prices are obtained. Finally, these results are used to determine the optimal financing policy of the firm, by solving a linear programming problem.

The method described above is based on Benders' Partitioning Procedure (14) for mixed-integer programs and can be broken up into a five-step scheme, where step 0 is the initiation of the process.

STEP 0.

Headquarters starts with an initial set of dual prices ρ_t^1 , $t = 1, \dots, T$, which satisfy the constraint set of (D), i.e. which belongs to the convex set formed by

$$\rho_t - \ell_t \rho_{t+1} \geq 0$$

$$(DCS) \quad -\rho_t - b_t \rho_{t+1} \geq 0$$

$$\rho_T = 1$$

$$\rho_t \geq 0.$$

This dual vector does not have to be a vertex of (DCS), it can be any feasible vector. The ρ_t^1 are passed out to the single departments, Step 1.

STEP 1.

Every single department evaluates its projects, based on the following criterion: they determine all those projects j for which

$$(CRIT) \quad \hat{a}_j + \sum_{t=1}^T \rho_t^{\ell} a_{tj} \cong 0, \quad \ell = 1, 2, \dots,$$

where the superscript ℓ indicates the ℓ -th iteration. The cash flow information (\hat{a}_j, a_{tj}) only about the acceptable projects, which satisfy (CRIT), is presented to headquarters, where it is incorporated in a corporate decision problem, Step 2.

STEP 2.

Headquarters formulates and solves the following programming problem

$$\begin{aligned}
 & \max_{x_j} x_o \\
 (CORP) \quad & x_o \cong \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \rho_t^1 (M_t + \sum_{j=1}^n a_{tj} x_j) \\
 & x_o \cong \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \rho_t^2 (M_t + \sum_{j=1}^n a_{tj} x_j) \\
 & \dots\dots\dots \\
 & x_o \cong \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \rho_t^{\ell} (M_t + \sum_{j=1}^n a_{tj} x_j).
 \end{aligned}$$

(CORP) is, in general, a mixed-integer program of a very special form--it contains only 0-1 variables x_j and one continuous variable x_o . The first iteration in the procedure requires the solution of (CORP) in the form

$$\begin{aligned}
 & \max_{x_j} x_o \\
 & x_o \cong \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \rho_t^1 (M_t + \sum_{j=1}^n a_{tj} x_j),
 \end{aligned}$$

which is a pure integer programming problem. Since only the projects with positive coefficients for x_j have been included the solution here is trivial. In general, Step 2 yields an over-all set of acceptable investment projects, x_j^{ℓ} , using the ρ_t^1 as dual evaluators, Step 3.

STEP 3.

Using the set of projects x_j^{ℓ} , that are acceptable to the corporation on the basis of the dual prices ρ_t^{ℓ} , headquarters forms and solves the following linear programming problem:

$$\min \sum_{t=1}^T \rho_t^1 (M_t + \sum_{j=1}^n a_{tj} x_j^{\ell})$$

(LIDU) subject to

$$\rho_t^{\ell} - \rho_{t+1}^{\ell} \geq 0, \quad t = 1, \dots, T-1$$

$$-\rho_t^{\ell} - \rho_{t+1}^{\ell} \geq 0, \quad t = 1, \dots, T-1$$

$$\rho_T^{\ell} = 1$$

$$\rho_t^{\ell} \geq 0.$$

(LIDU) is obtained from (D), the max-min type dual of (P), by using the fixed decision vector x_j^1 . Let the solution to (LIDU) be $\rho_t^{\ell+1}$. This set of dual prices is passed out to the single divisions, where new evaluations of the available investment opportunities are made, using criterion (CRIT) from Step 1. The cash flows associated with all acceptable projects $x_j^{\ell+1}$ are presented to headquarters. Headquarters conducts the following test: it determines whether or not

$$(TEST) \quad x_0 - \sum_{j=1}^n \hat{a}_j x_j \cong \sum_{t=1}^T \rho_t^{\ell+1} (M_t + \sum_{j=1}^n a_{tj} x_j)$$

holds. If the equality in (TEST) is satisfied, Step 4. If it is not satisfied, go to Step 2 and add

$$x_0 \cong \sum_{j=1}^n \hat{a}_j x_j + \sum_{t=1}^T \rho_t^{\ell+1} (M_t + \sum_{j=1}^n a_{tj} x_j)$$

to the existing set of (CORP). Then headquarters solves this new problem and goes back to Step 3. An inequality in the constraint set of (CORP) can be dropped if the corresponding slack becomes positive.

STEP 4.

Now, use is made of the optimal "over-all" set of investments, \bar{x}_j , to formulate and solve the linear program

$$\max v_T - w_T$$

(FIN) subject to

$$\begin{aligned} -\rho_{t-1} v_{t-1} + v_t + b_{t-1} w_{t-1} - w_t &\cong \\ M_t + \sum_{j=1}^n a_{tj} \bar{x}_j, \quad t = 1, \dots, T \\ v_t, w_t &\cong 0, \quad \text{all } t. \end{aligned}$$

Let the solution to (FIN) be (\bar{v}_t, \bar{w}_t) , all t . This solution then provides the optimal financing policy for the company.

Claim: A project will be accepted only if

$$\hat{a}_j + \sum_{t=1}^T a_{tj} \rho_t^{\ell} = 0,$$

at least for one ℓ .

Proof.

Let us assume the contrary, i.e. that there is an optimal solution with one or more projects k such that

$$\hat{a}_k + \sum_{t=1}^T a_{tk} \rho_t^\ell < 0, \text{ for all } \ell.$$

Then these projects cause nonpositive contributions

$$\sum_k \left(\hat{a}_k + \sum_{t=1}^T a_{tk} \rho_t^\ell \right)$$

of various size in all of the ℓ constraints of (CORP); therefore, to set the corresponding x_k equal to zero, would cause a monotonous increase of the values of all right-hand sides in (CORP), including those active constraints that currently determine the value of x_0 . It follows that the considered solution could not be optimal, as assumed in the hypothesis. Q.e.d.

We see that this 5-step procedure provides a method of determining a firm's investment and financing policy, with a considerable part of the decision process being delegated from headquarters to the single departments of the organization. At the same time, a set of internally generated discount rates is provided, thus solving the problem of determining the cost of capital.

CHAPTER V

BAUMOL AND QUANDT'S UTILITY MODEL

5.1 Introduction

The main difficulty in the application of the Basic Model is the determination of the present value of project j ; in particular, which discounting factors should be used for computing b_j ? Similar questions arise for the Terminal Wealth Model when the \hat{a}_j have to be determined. Lorie and Savage (14) and Weingartner (15) all mentioned the difficulties but nothing to resolve them; they assumed that the appropriate discount rates were available. Baumol and Quandt (5) attempted unsuccessfully to establish appropriate discounting factors internally by utilizing the dual model. They pointed out that in models, where the firm operates without resources to a capital market, it is cut off from any external discount criteria and thus external rates of interest are irrelevant.

Baumol and Quandt showed that it is impossible to use simultaneously a present value formulation of the objective function and to have the relevant discount rates generated by the model. Their approach was to turn to a formulation using an investor's utility function. As in the preceding chapters, we will introduce their model first in its original form as a linear programming problem and then analyze the effects of introducing indivisibilities, making use of Balas's duality and its economic implications.

5.2 The Linear Utility Model

The problem under consideration is to find optimal investment

and dividend policies for a firm without resource to a capital market. The firm considers n different, clearly defined investment projects, its capital expenditures are limited. The investor is taken to wish ultimately to maximize--as Baumol and Quandt put it--his ability to consume, i.e. the sum of withdrawals of cash made available by the projects undertaken, each withdrawal weighted by its subjective utility.

Letting

W_t be the cash dividend to be paid to the owner at time t ;

U_t be the fixed utility of a dollar in period t ;

a_{tj} be the net cash flow obtained from a unit of project j during period t ;

M_t be the amount of cash available from projects outside the analysis and from other sources at time t ;

x_j be the number of units of project j invested;

and assuming a planning horizon of T years, the mathematical statement of the problem is:

$$\max \sum_{t=1}^T U_t W_t$$

(LP) subject to

$$- \sum_{j=1}^n a_{tj} x_j + W_t \leq M_t, \quad t = 1, \dots, T$$

$$x_j, W_t \geq 0.$$

Baumol and Quandt assumed that utility is nonnegative and linear in money, although they pointed out the possibility of a nonlinear

objective function. The structural constraints in (LP) were shown (5) to hold as strict equalities, since every additional dollar would be paid out as a dividend at a positive utility. As opposed to Weingartner's Basic Model, (LP) does consider the owner's time preferences for consumption.

Some of the drawbacks of the model are that it assumes an identity between the firm and the single owner-entrepreneur,¹ and the fact that it requires an assignment of the utility measure in advance of information about the withdrawal possibilities. Another problem arises through the fact that the model does not consider post horizon cash flows from projects accepted. This may change the overall desirability of these projects and it also makes the solution sensitive to the choice of the horizon. It, in fact, means in this model, where investments have constant returns to scale without limit, that the maximum number of projects selected is T, the number of periods to the horizon.

5.3 The Dual Problem

The linear programming problem (LP) has a dual of the form

$$\begin{aligned}
 & \min \sum_{t=1}^T \rho_t M_t \\
 & \text{subject to} \\
 & - \sum_{t=1}^T a_{tj} \rho_t \cong 0, \quad j = 1, \dots, n
 \end{aligned}
 \tag{LD}$$

¹ or it assumes that all stockholders have the same time preferences.

$$\rho_t \equiv U_t, \quad t = 1, \dots, T$$

$$\rho_t \equiv 0.$$

To interpret the dual variables ρ_t , consider an optimal solution and assume that \bar{w}_{t_1} and \bar{w}_{t_2} are positive dividend payments in the periods t_1 and t_2 in this optimal solution. Then the corresponding dual constraints must be strict equalities:

$$\bar{\rho}_{t_1} = U_{t_1}$$

$$\bar{\rho}_{t_2} = U_{t_2}$$

so that

$$\frac{U_{t_1}}{U_{t_2}} = \frac{\bar{\rho}_{t_1}}{\bar{\rho}_{t_2}}.$$

Baumol and Quandt interpret the last equation as follows: the marginal rate of substitution between withdrawals in the two periods equals the discount rate at the optimum if funds are withdrawn during both period t_1 and period t_2 . U_{t_1}/U_{t_2} is to be understood as a relative, subjective discount rate, whereas the "objective rate" ρ_{t_1}/ρ_{t_2} is based on the actual production opportunities available.

Suppose, at optimality no money is withdrawn in period t_3 , then this may be interpreted as the reasonable statement that $\bar{\rho}_{t_3}$, the marginal objective utility of \$1 at time t must be at least equal to the subjective utility U_{t_3} .

After this brief introduction, we consider Baumol and Quandt's model under the additional assumption that the projects are indivisible in nature, and that multiple projects are excluded.

5.4 The Mixed-Integer Programming Formulation

Adding the assumption of indivisible projects by means of a 0-1 restriction on the x_j , we obtain the following model

$$\max \sum_{t=1}^T U_t W_t \quad (5.1)$$

subject to

$$- \sum_{j=1}^n a_{tj} x_j + W_t \leq M_t, \quad \text{all } t \quad (5.2)$$

$$x_j = 0 \text{ or } 1, \quad \text{all } j$$

$$W_t \geq 0, \quad \text{all } t.$$

(P) represents a mixed-integer programming problem, corresponding to case (a) in Balas's qualification, since the set of dual integer variables is empty. The model produces the optimal set of indivisible projects and the optimal combination of dividend payments up to the horizon time. As in the case of the linear programming formulation, the constraints of (P) must always be equalities in an optimal solution, since the subjective utilities U_t are assumed to be strictly positive for all t .

According to Balas's duality, we can construct the dual of (P)

$$\max_{x_j} \min_{\rho_t} \sum_{t=1}^T \rho_t M_t - \sum_{j=1}^n \mu_j x_j \quad (5.3)$$

(D) subject to

$$-\sum_{t=1}^T a_{tj} \rho_t - \mu_j = 0, \quad \text{all } j \quad (5.4)$$

$$\rho_t - \lambda_t = U_t, \quad \text{all } t \quad (5.5)$$

$$\rho_t, \lambda_t \geq 0, \quad \text{all } t$$

$$x_j = 0 \text{ or } 1, \quad \text{all } j$$

$$\mu_j \text{ unrestricted.}$$

The dual problem (D) is a constrained mixed-integer optimization problem of the type max-min. The nonnegative dual surplus variables, λ_t , correspond to the continuous primal variables, W_t , whereas the unrestricted dual surplus variables, μ_j , correspond to the 0-1 constrained primal variables, x_j . As in the case of any mixed-integer programming problem and its dual, the coefficient submatrix $A^{12} = \emptyset$ (in Balas's notation), and the constraint qualification is met.

5.5 Duality Relations and the Analysis

of Primal and Dual at Optimality

As in the former chapters, we are able to state and derive several properties of (P) and (D) which deviate distinctly from the case of the linear formulations (LP) and (LD). The first property we state, however, will trivially hold for both cases.

If M_t is nonnegative for $t = 1, \dots, T$, then (P) has always feasible solutions. Obviously, here $x_j \equiv 0$, all j , and $W_t \equiv 0$, all t , will always be feasible to (P).

Applying Balas's Theorem 1, we find that primal and dual are involutory.

Theorem 2, which holds in accordance with the fulfilled constraint qualification, here reads as follows: If (P) has an optimal solution $(\bar{x}_1, \dots, \bar{x}_n; \bar{W}_1, \dots, \bar{W}_T)$, there exists a set of dual variables $(\bar{\rho}_1, \dots, \bar{\rho}_T)$, such that $(\bar{\rho}_1, \dots, \bar{\rho}_T; \bar{x}_1, \dots, \bar{x}_n)$ is an optimal solution to (D) with

$$\max \sum_{t=1}^T U_t \bar{W}_t = \max_{x_j} \min_{\rho_t} \sum_{t=1}^T \bar{\rho}_t M_t + \sum_{j=1}^n \sum_{t=1}^T a_{tj} \bar{\rho}_t, \quad (5.6)$$

i.e., the maximum sum of cash dividends, each dividend weighted by its subjective utility, is equal to the value of the outside resources, evaluated at the dual variables $\bar{\rho}_t$, minus the value of the cash flows using the same evaluators.

Furthermore, the function

$$\begin{aligned} F(x_j, W_t, \rho_t) &= \sum_{t=1}^T U_t W_t + \sum_{t=1}^T \rho_t M_t + \sum_{t=1}^T \rho_t \\ &\quad \cdot \left(\sum_{j=1}^n a_{tj} x_j - W_t \right) = \\ &= \sum_{t=1}^T U_t W_t + \sum_{t=1}^T \rho_t y_t \end{aligned}$$

has a saddle-point at $(\bar{x}_1, \dots, \bar{x}_n; \bar{W}_1, \dots, \bar{W}_T; \bar{\rho}_1, \dots, \bar{\rho}_T)$:

$$\begin{aligned}
\sum_{t=1}^T U_t W_t + \sum_{t=1}^T \bar{\rho}_t y_t &\cong \sum_{t=1}^T U_t \bar{W}_t \cong \\
&\cong \sum_{t=1}^T \rho_t M_t - \sum_{j=1}^n \mu_j \bar{x}_j,
\end{aligned} \tag{5.7}$$

for all $(x_j, W_t) \in X(\bar{y}_t)$, and all $\rho_t \in U(\bar{\lambda}_t)$. Here, $X(\bar{y}_t)$ is the set of all primal variables satisfying the primal constraint set, when the primal slack vector is fixed at its optimal level (\bar{y}_t) , and $U(\bar{\lambda}_t)$ is the set of all dual variables ρ_t satisfying the dual constraints (5.5), when the dual surplus vector (that corresponds to the continuous primal variables) is fixed at its optimal level $(\bar{\lambda}_t)$.

The next result, obtained from Theorem 4, is: If $(\bar{x}_1, \dots, \bar{x}_n; \bar{W}_1, \dots, \bar{W}_T)$ and $(\bar{\rho}_1, \dots, \bar{\rho}_T; \bar{x}_1, \dots, \bar{x}_n)$ are optimal solutions to (P) and (D) respectively, then the following complementary slackness conditions hold:

$$\bar{\rho}_t \cdot \bar{y}_t = 0, \quad t = 1, \dots, T \tag{5.8}$$

$$\bar{W}_t \cdot \bar{\lambda}_t = 0, \quad t = 1, \dots, T \tag{5.9}$$

$$\sum_{t=1}^T U_t \bar{W}_t - \sum_{t=1}^T \bar{\rho}_t (M_t + \sum_{j=1}^n a_{tj} \bar{x}_j) = 0. \tag{5.10}$$

From the dual constraints (5.5) we know that $\rho_t \cong U_t > 0$, for all t , what, together with (5.8), implies $\bar{y}_t = 0$, for all t , as stated before. Condition (5.9) is equivalent to the statement that, at optimality, dividends are paid out only in those periods where the associated subjective utilities U_t are equal to the corresponding dual variables $\bar{\rho}_t$.

Condition (5.10) turns out to be identical to the symmetry relationship (5.6), if we substitute for the μ_j from the constraints (5.4).

Finally, the existence of a unique nondegenerate optimal solution to (P) implies the existence of a unique optimal solution to (D), according to Theorem 5.

5.6 The Equivalent Linear Program and its Economic Interpretation

So far, we did not use the concepts of marginality in the interpretation of the dual variables ρ_t , although we know that their dimension must be that of utilities. The mixed-integer program (P) can be solved, applying, for instance, Benders' Partitioning Procedure, and yields an optimal solution $(\bar{x}_1, \dots, \bar{x}_n; \bar{w}_1, \dots, \bar{w}_T)$. This solution, or more precisely, its binary components \bar{x}_j , can be used to solve (D), which becomes a linear program. Then, by means of Theorem 6, the following linear program (ELP) can be constructed:

$$\begin{aligned}
 & \max \sum_{j=1}^n s_j x_j + \sum_{t=1}^T U_t w_t \\
 \text{(ELP)} \quad & \text{subject to} \\
 & - \sum_{j=1}^n a_{tj} x_j + w_t \leq M_t, \quad \text{all } t \\
 & x_j \leq 1, \quad \text{all } j \\
 & x_j, w_t \geq 0, \quad \text{all } j, \text{ all } t,
 \end{aligned}$$

where quantities s_j are given by

$$s_j = \bar{\mu}_j, \quad \text{if } \bar{x}_j > 0,$$

$$s_j = \min(0, \bar{\mu}_j), \quad \text{if } \bar{x}_j = 0,$$

with

$$\bar{\mu}_j = - \sum_{t=1}^T a_{tj} \bar{\rho}_t \quad (\text{unrestricted}).$$

The dual of (ELP) has the form

$$\min \sum_{t=1}^T \rho_t M_t + \sum_{j=1}^n w_j$$

(ELD)

subject to

$$- \sum_{t=1}^T a_{tj} \rho_t + w_j \cong s_j, \quad \text{all } j$$

$$\rho_t \cong U_t, \quad \text{all } t$$

$$\rho_t, w_j \cong 0, \quad \text{all } t, \text{ all } j.$$

Let us analyze this pair of linear programs in more detail.

Whereas in the case of Weingartner's Basic Model, the quantities s_j played the role of subsidies or penalties for certain projects, which were originally characterized by their unit profits c_j , the objective function in Baumol and Quandt's model does not contain such profits c_j at all. Therefore, the results of the linearization of (P) in the above manner are not only the introduction of the variables w_j , caused by the upper bound constraints on the x_j , but also a conceptually

different objective function. The objective function in (ELP) contains, besides the term in the utilities of the dividend payments, a term involving "profits" and "losses"--in a cost accounting sense--associated with each project j .

Similar to the case of the Basic Model, the quantities w_j will be seen to play the role of project evaluators; at optimality, they are natural by-products of the linear programming solution for ranking all projects considered. Their value is an indicator for the desirability of a project or the economic goodwill associated with the return from that project.

Now operating in a linear programming framework, we are able to use the concepts of shadow prices as evaluators of the budget resources. Due to the results of Chapter II, the properties of (ELP) and (ELD), as far as they are relevant for an economic interpretation of the model, are:

If (P) has an optimal solution $(\bar{x}_1, \dots, \bar{x}_n; \bar{w}_1, \dots, \bar{w}_T)$, there exists a vector of utilities $\bar{\rho}_t \geq 0$, all t , and quantities s_j , that are unrestricted in sign, such that

(a) an optimal solution to (P) is also optimal to (ELP);

(b) $-\sum_{t=1}^T a_{tj} \bar{\rho}_t + \bar{w}_j \geq s_j$, i.e.

the sum of discounted outlays, as evaluated at $\bar{\rho}_t$, the marginal objective utility of each budget year, plus the optimal value of the corresponding

project evaluators, \bar{w}_j , is never less than the imputed profit s_j , associated with project j ;

$$(c) \bar{\rho}_t \geq U_t, \text{ i.e.}$$

the marginal objective utility of \$1 at time t must be at least equal to the subjective utility U_t , procurable from its possible use as a dividend payment at that time;

$$(d) \sum_{j=1}^n \left(- \sum_{t=1}^T a_{tj} \bar{\rho}_t + \bar{w}_j - s_j \right) \bar{x}_j = 0,$$

what implies, considering the nonnegativity of all terms in the sum,

$$\left(- \sum_{t=1}^T a_{tj} \bar{\rho}_t + \bar{w}_j - s_j \right) \bar{x}_j = 0, \text{ i.e.}$$

a project will be accepted only if the corresponding constraints in (ELD) hold as an equality. Then, the "value" of an accepted project, \bar{w}_j , will be the excess, if any, of its recomputed profit s_j over the sum of outlays as evaluated at $\bar{\rho}_t$;

$$(e) \sum_{t=1}^T \bar{\rho}_t (M_t + \sum_{j=1}^n a_{tj} \bar{x}_j - \bar{w}_t) = 0, \text{ or,}$$

by the same argument as in (d),

$$\bar{\rho}_t (M_t + \sum_{j=1}^n a_{tj} \bar{x}_j - \bar{w}_t) = 0, \text{ and, since all objective}$$

utilities $\bar{\rho}_t$ are strictly positive ($\bar{\rho}_t \geq U_t > 0$),

$$M_t + \sum_{j=1}^n a_{tj} \bar{x}_j - \bar{w}_t = 0, \text{ i.e.}$$

at optimality, all budgets are exhausted, since every additional dollar would be paid out as dividend at a positive utility;

$$(f) \sum_{j=1}^n \bar{w}_j (1 - \bar{x}_j) = 0, \text{ or, for each single project,}$$

$$\bar{w}_j (1 - \bar{x}_j) = 0, \text{ i.e.}$$

project j has to be accepted if the value of its indicator \bar{w}_j is positive, in the optimal solution; it may or may not be acceptable if this value is zero;

$$(g) \sum_{j=1}^n s_j \bar{x}_j + \sum_{t=1}^T u_t \bar{w}_t = \sum_{t=1}^T \bar{p}_t M_t + \sum_{j=1}^n \bar{w}_j ,$$

i.e. the imputed profit from all accepted projects plus the sum of all dividends, weighted with their subjective utilities, must be, for the optimal case, equal to the sum of all budgets, evaluated at the corresponding objective utilities, plus the total "value" of all accepted projects.

CHAPTER VI

SYNTHESIS OF A NEW MODEL

6.1 Introduction

There seems to be a rather widespread agreement in the literature on the mathematics of finance that an appropriate objective in the planning of a firm's productive investment and financing policy is the maximization of some function, usually a discounted sum, of all anticipated dividend payments to the owners of the firm's present shares. If, in the function's argument, the stream of dividends is truncated at some finite horizon time, T , as required in a programming formulation, then it seems reasonable to include also in the argument the time T terminal wealth as an indicator for the post T stream of dividends.

The models we discussed in the Chapters III and IV set the total profit and the terminal wealth, respectively, as the objective to be maximized, and simply did not allow dividend payments prior to or at the horizon time. The Baumol-Quandt Model, on the other hand, was using a utility formulation involving dividend payments in period $1, \dots, T$, but showed a certain lack of realism because of the extremely simple structure of its constraints.

In this chapter, we construct a model that represents a synthesis of both approaches discussed before, by maximizing a function of a finite stream, W_1, \dots, W_T , of dividend payments and of the terminal wealth of the firm under consideration. A further market imperfection, in addition to the difference between borrowing and lending rates, is introduced in form

of upper bounds on the amounts borrowed. The solution to this model yields the optimal investment, dividend, and financing policy of the firm under consideration. Rather than assuming constant returns-to-scale for the available investment projects, we assume that the projects are indivisible in nature.

6.2 Statement of the Problem

A firm considers n different, clearly defined investment projects. The firm is able to interact with the capital market where it can lend and borrow money at predetermined discount rates. The capital expenditures as well as the amount of money borrowed in a certain period are limited. The firm's objective is to maximize the discounted stream of dividend payments plus the terminal wealth of the company as of a finite time horizon T .

Let

- a_{tj} be the net cash flow from the j -th project in period t ;
- \hat{a}_j be the horizon time T present value of post T cash flows from project j ;
- M_t be the amount of cash made available from projects outside the analysis and from other outside sources at time t ;
- ℓ_t be equal to $1 + r_{\ell t}$, where $r_{\ell t}$ is the lending rate of interest from time t to $t + 1$;
- b_t be equal to $1 - r_{bt}$, where r_{bt} is the borrowing rate of interest from time t to $t + 1$; it is assumed that $b_t \geq \ell_t$;
- B_t be the maximum value of w_t , where w_t is a variable defined below, at time t ;

x_j be a binary variable with the value 1 if the j -th project is adopted and 0 otherwise;

W_t be the dividend to be paid at time t ;

p_t be the rate at which such dividends are valued by the stockholders; p_T is assumed to be equal to 1;

w_t be the cash to be borrowed from time t to $t + 1$;

v_t be the cash to be lent from time t to $t + 1$; $v_0 = w_0 \equiv 0$, by definition.

Choosing the horizon time T as the reference point for all considerations concerning the "time value of money," the mathematical statement of the problem is:

$$\max p_T \left(\sum_{j=1}^n \hat{a}_j x_j + v_T - w_T \right) + \sum_{t=1}^T p_t W_t \quad (6.1)$$

(P) subject to

$$- \sum_{j=1}^n a_{tj} x_j - \ell_{t-1} v_{t-1} + v_t + b_{t-1} w_{t-1} + w_t + W_t \leq M_t, \quad \text{for } t = 1, \dots, T \quad (6.2)$$

$$w_t \leq B_t, \quad \text{for } t = 1, \dots, T - 1 \quad (6.3)$$

$$x_j = 0 \text{ or } 1, \quad \text{for } j = 1, \dots, n$$

$$v_t, w_t, W_t \geq 0, \quad \text{all } t.$$

The objective function (6.1) contains basically three components,

(a) the net amount of all financial assets accumulated at the horizon,

$v_T - w_T$, (b) the time T present value of post T cash flows, $\sum_{j=1}^n \hat{a}_j x_j$,

and (c) the sum of all dividend payments up to the horizon time, all terms

weighted with the respective rates p_t at which such dividends are valued by the stockholders. We shall assume in this model that $p_t > 0$ for all t . Consequently, in the objective function the coefficient of W_t will be greater than zero, and therefore any slack would clearly be paid out as dividend.

Using the assumption $p_T \equiv 1$ and introducing slack variables into (P), the mathematical programming model may be written as

$$\begin{aligned}
 & \max \sum_{j=1}^n \hat{a}_j x_j + v_T - w_T + \sum_{t=1}^T p_t W_t \\
 (P) \quad & \text{subject to} \\
 & - \sum_{j=1}^n a_{tj} x_j - \ell_{t-1} v_{t-1} + b_{t-1} W_{t-1} + w_t + W_t + y_t = M_t, \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{all } t \\
 & w_t + q_t = B_t, \qquad \qquad \qquad \text{for } t = 1, \dots, T-1 \\
 & x_j = 0 \text{ or } 1, \qquad \qquad \qquad \text{all } j \\
 & v_t, w_t, W_t \geq 0.
 \end{aligned}$$

The cash balance restriction (6.2), taking the terms in order, say that the net cash outflow to projects, minus the cash inflow from time $t-1$ loans, plus the cash outflow for time t loans, plus the cash outflow for time t dividends, must be less or equal to the cash available from outside sources at time t . The amounts from outside sources, M_t , will be assumed to be nonnegative for all t up to the horizon time.

6.3 Problems Caused by the Finite Horizon

Before we formulate the dual problem, we want to use this model as an example for some of the problems arising when the assumption of a finite horizon is made. Consider $\sum_{j=1}^n \hat{a}_j x_j$, the time T present value of post T cash flows. If for periods beginning at $T + 1$, each period has one single market rate, r_t , at which borrowing and lending may take place, then future cash flows can without problem be converted to a time T present worth

$$\hat{a}_j = \sum_{t=T+1}^{\infty} a_{tj} \prod_{k=T+1}^t \lambda_k^{-1}, \quad \text{with } \lambda_t = 1 + r_t.$$

But let us now consider the case where $b_T > \ell_T$, and let d be a cash receipt which is to occur at time $T + 1$. Then the time T value of d depends on whether the firm is a borrower or a lender (or either) from T to $T + 1$. If, e.g., the firm is a borrower, then at T , it may borrow $d \cdot b_T^{-1}$ and pay it off at $T + 1$ with the d . Hence, we should say that d , at $T + 1$, is worth $d \cdot b_T^{-1}$ at T .

If, on the other hand, the firm is a lender from T to $T + 1$, then availability of the d at $T + 1$ relieves us of lending an amount $d \cdot \ell_T^{-1}$ at T . In this case, we would say that d at $T + 1$ has a time T value of $d \cdot \ell_T^{-1} > d \cdot b_T^{-1}$. Therefore, just what a post T cash flow is worth at T cannot be determined until we know the firm's post T borrowing and lending pattern. Hence, we will assume for our model that each post T period has a single market rate, r_t , for borrowing and lending.

Closely connected with the above arguments is the explanation for the fact that no upper bound on borrowing was imposed for the time period

T. Suppose, for example, $B_T = 0$. Then \bar{w}_T , the optimal value, has to be zero, too. What then is the time $T + 1$ cash flow, d , worth at T ? If it saves some lending, it is again worth $d \cdot \ell_T^{-1}$. But, if it is needed at T (if the firm is a prospective borrower), the d cannot be converted to time point T . Again, we see that the time T value of d depends on the post T financing picture, which is a product of the analysis, rather than being given a priori as some given constant times the d . Hence, if we wish to measure a_j in advance, we must avoid to have a constraint

$$w_T \leq B_T.$$

What is the optimal choice for a horizon time T ? With the assumption of just a single market rate, r_t , for each post T period, T ought to be large to make the model more realistic. However, this aspect must be weighted against the increased number of constraints and hence probably increased difficulty of solution, which a larger value of T would imply, as well as the increased difficulty to obtain sufficiently reliable estimates (forecasts) for the various cash flows to be explicitly considered in the model.

6.4 The Dual Problem

According to the duality concepts stated in Chapter II the dual of (P), a mixed-integer programming problem, is a max-min type optimization problem of the form

$$\max_{x_j} \min_{\rho_t, \beta_t} \sum_{t=1}^T \rho_t M_t + \sum_{t=1}^T \beta_t B_t - \sum_{j=1}^n \mu_j x_j \quad (6.4)$$

(D) subject to

$$-\sum_{t=1}^T a_{tj} \rho_t - \mu_j = \hat{a}_j, \quad \text{all } j \quad (6.5)$$

$$\rho_t - \ell_t \rho_{t+1} \geq 0, \quad t = 1, \dots, T-1 \quad (6.6)$$

$$-\rho_t + b_t \rho_{t+1} + \beta_t \geq 0, \quad t = 1, \dots, T-1 \quad (6.7)$$

$$\rho_T \geq 1 \quad (6.8)$$

$$-\rho_T \geq -1 \quad (6.9)$$

$$\rho_t - \lambda_t = p_t \quad (6.10)$$

$$\rho_t, \beta_t, \lambda_t = 0, \quad \text{all } t$$

$$\mu_j \text{ unrestricted, all } j.$$

Thus, the dual contains the binary primal variables x_j ; the continuous dual variables ρ_t, β_t ; the nonnegative dual surplus variables λ_t , and (6.6) and (6.7), all corresponding to the continuous primal variables, W_t, v_t , and w_t ; and, finally the unrestricted dual surplus variables μ_j associated with the discrete primal x_j .

Table 6.1 shows the relationships between all variables and constants occurring in (P) and (D), where both problems are stated in terms of the original set of inequalities.

	$x_1 \dots x_n$	$v_1 \dots v_T$	$w_1 \dots w_T$	$W_1 \dots W_T$	
ρ_1	$-a_{11} \dots -a_{1n}$	1	-1	1	$\cong M_1$
.	.	$-1_1 \ 1$	$b_1 \ -1$	1	.
.
.	.	1	-1	.	.
ρ_T	$-a_{T1} \dots -a_{Tn}$	$-1_{T-1} \ 1$	$b_{T-1} \ -1$	1	$\cong M_T$
β_1			1		$\cong B_1$
.			1		.
.	\emptyset	\emptyset	1	\emptyset	.
.			1		.
β_T			0		$\cong B_T$
	$\cong \hat{a}_1 \dots \cong \hat{a}_n$	$\cong 0 \dots \cong 0 \cong 1$	$\cong 0 \dots \cong 0 \cong -1$	$\cong p_1 \dots \cong p_T$	

Table 1. Tabular Representation of Primal and Dual Problems

Using the notational correspondence

<u>New Model</u>	<u>Balas</u>
(x_1, \dots, x_n)	x^1
$(v_1, \dots, v_T; w_1, \dots, w_T; W_1, \dots, W_T)$	x^2
$(\rho_1, \dots, \rho_T; \beta_1, \dots, \beta_T)$	u^2
$(M_1, \dots, M_T; B_1, \dots, B_{T-1}, 0)$	b^2
(a_1, \dots, a_n)	c^1
$(0, \dots, 1; 0, \dots, -1; p_1, \dots, p_T)$	c^2

(P) and (D) read in Balas's notation

$$\max c^1 x^1 + c^2 x^2$$

(P') subject to

$$A^{21} x^1 + A^{22} x^2 \leq b^2$$

$$x_j = 0 \text{ or } 1, \quad j \in N_1$$

$$x_j = 0, \quad j \in N - N_1, \text{ and}$$

$$\max_{x^1} \min_{u^2} u^2 b^2 - v^1 x^1$$

(D') subject to

$$u^2 A^{21} - v^1 = c^1$$

$$u^2 A^{22} - v^2 = c^2$$

$$x_j = 0 \text{ or } 1, \quad j \in N_1$$

$$v_j \text{ unrestricted, } j \in N_1$$

$$u_1, v_j \geq 0, \quad i \in M - M_1, \quad j \in N - N_1.$$

Since $A^{12} = \emptyset$, the constraint qualification is met, so that Theorem 2 will apply.

6.5 Properties of the Model and Economic Interpretations

The capital budgeting model (P) under consideration becomes equivalent (not identical) to Weingartner's Terminal Wealth Model, if we assure $p_t \equiv 0$, all t , and increase B_t , all t , without limit; then, by complementary slackness, W_t will always be zero at optimality, and the optimal solutions will be equal for both problems. On the other hand, we arrive at the Baumol-Quandt Model if we exclude upper bounds on borrowing, borrowing and lending transactions themselves, and the terminal wealth, as part of the maximizing objective, from our model.

The theorems in Chapter II can be applied and lead to the following results:

Theorem 1 (Involution) holds in the same form as stated there.

The vector of primal slack variables $(y_1, \dots, y_T; q_1, \dots, q_T)$ is componentwise separable with respect to the vector of the discrete primal decision variables (x_1, \dots, x_n) , and therefore Theorem 2 holds:

If (P) has an optimal solution $(\bar{x}_1, \dots, \bar{x}_n; \bar{v}_1, \dots, \bar{v}_T; \bar{w}_1, \dots, \bar{w}_T; \bar{w}_1, \dots, \bar{w}_T)$, there exist sets of dual variables, $(\bar{\rho}_1, \dots, \bar{\rho}_T)$ and $(\bar{\beta}_1, \dots, \bar{\beta}_T)$ such that $(\bar{\rho}_1, \dots, \bar{\rho}_T; \bar{\beta}_1, \dots, \bar{\beta}_T; \bar{x}_1, \dots, \bar{x}_n)$ is an optimal solution to (D), and the symmetry relationship

$$\begin{aligned}
\max \sum_{j=1}^n a_j x_j + v_T - w_T + \sum_{t=1}^T p_t W_t \\
= \max_{x_j} \min_{\rho_t, \beta_t} \sum_{t=1}^T \rho_t M_t + \sum_{t=1}^T \beta_t B_t - \sum_{j=1}^n \mu_j x_j
\end{aligned} \tag{6.11}$$

holds. We will interpret this equality after the analysis of the single variables involved.

In accordance with Theorem 4, we are able to state the following complementary slackness conditions:

$$\bar{y}_t \cdot \bar{\rho}_t = 0, \quad t = 1, \dots, T \tag{6.12}$$

$$\bar{q}_t \cdot \bar{\beta}_t = 0, \quad t = 1, \dots, T - 1 \tag{6.13}$$

$$(\bar{\rho}_t - \ell_t \bar{\rho}_{t+1}) \bar{v}_t = 0, \quad t = 1, \dots, T - 1 \tag{6.14}$$

$$(-\bar{\rho}_t + b_t \bar{\rho}_{t+1}) \bar{w}_t = 0, \quad t = 1, \dots, T - 1 \tag{6.15}$$

$$\bar{\lambda}_t \cdot \bar{w}_t = 0, \quad t = 1, \dots, T \tag{6.16}$$

$$v_T - w_T - \sum_{t=1}^T \bar{\rho}_t M_t - \sum_{t=1}^T \bar{\beta}_t B_t + \sum_{t=1}^T (\bar{\rho}_t + \bar{\beta}_t) \sum_{j=1}^n a_{tj} \bar{x}_j = 0, \tag{6.17}$$

where y_t stands for the excess funds in period t , q_t for the unused amount available for borrowing, and $\lambda_t = \rho_t - p_t$.

Using the relationships stated above and the constraint sets of (P) and (D), we are now able to derive a number of properties of the various problem variables, which will give more insight into the structure of the model.

Lemma 5. In an optimal solution to (P) and (D) the following relationships hold:

- (a) $\bar{\rho}_t > 0$, all t
- (b) $\bar{\rho}_T = 1$
- (c) $\bar{y}_t = 0$, all t

Proof.

(b) follows directly from (6.8) and (6.9);

(a) here, we can rewrite (6.6) in the form

$$\bar{\rho}_t = \ell_t \bar{\rho}_{t+1},$$

and, by repeated application of this inequality, we obtain

$$\bar{\rho}_t = \left(\prod_{r=t}^{T-1} \ell_r \right) \bar{\rho}_T = \prod_{r=t}^{T-1} \ell_r > 0;$$

(c) follows from (a) together with the complementary slackness condition (6.12). Q.e.d.

Lemma 6. Assume that $l_t < b_t$. Then

$$\bar{w}_t \cdot v_t = 0, \text{ for all } t;$$

i.e., the firm will never borrow and lend money in the same period.

Proof.

Combining the constraints (6.6) and (6.7), we obtain

$$\ell_t \bar{\rho}_{t+1} \leq \bar{\rho}_t \leq b_t \bar{\rho}_{t+1} + \bar{\beta}_t. \quad (6.18)$$

By complementary slackness (6.13) and (6.14), if $\bar{v}_t > 0$, $\ell_t \bar{\rho}_{t+1} = \bar{\rho}_t$ and $\bar{w}_t = 0$ holds; i.e. from t to $t+1$ the firm is lending money, and the ratio of two successive dual variables, $\bar{\rho}_t$ to $\bar{\rho}_{t+1}$, is the lending rate, ℓ_t . Similarly, if the firm is a borrower from t to $t+1$, i.e. if $\bar{w}_t > 0$, then $\bar{\rho}_t = b_t \bar{\rho}_{t+1} + \bar{\beta}_t$, and $\bar{v}_t = 0$ holds.

In this case, the ratio of $\bar{\rho}_t$ to $\bar{\rho}_{t+1}$ is equal to the lending rate only if $\bar{w}_t < B_t$, what implies that $\bar{\beta}_t = 0$.

If, for period t , both \bar{w}_t and \bar{v}_t are equal to zero, then all we know is that the general expression (6.18) holds. Q.e.d.

Let us analyze now the special case $\ell_t = b_t = r_t$. Then (6.18) becomes

$$r_t \bar{\rho}_{t+1} \cong \bar{\rho}_t \cong r_t \bar{\rho}_{t+1} + \bar{\beta}_t \quad . \quad (6.19)$$

Assume that $\bar{w}_t > 0$; then $\bar{\rho}_t = r_t \bar{\rho}_{t+1} + \bar{\beta}_t$, and two cases are possible:

(a) if $0 < \bar{w}_t < B_t$; $\bar{\beta}_t = 0$ and $\bar{v}_t \cong 0$; i.e. if the upper bound constraint on the amount borrowed is not active, we are indifferent between lending and borrowing in this period; it may be optimal to do both;

(b) if $\bar{w}_t = B_t$, then $\bar{\beta}_t > 0$ implies $\bar{v}_t = 0$, i.e. in a period where it is optimal to borrow as much as possible in the capital market, it will not be optimal to lend away money at the same time.

Let us finally ask the question in which period dividends are paid to the shareholders. The answer is obtained from the complementary slackness condition (6.13)

$$\bar{w}_t (\bar{\rho}_t - p_t) = 0.$$

This condition implies that dividends are paid in period t only if the value of the corresponding dual variable $\bar{\rho}_t$ is equal to p_t , the rate at which dividends in period t are valued by the shareholders. If,

however, $\bar{\rho}_t > p_t$, then maximizing the overall objective asks for utilizing available funds for financing projects and/or lending cash in period t .

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In the development of this research, a duality concept due to Balas for discrete programming was used to handle the mathematical aspects of four capital budgeting models, which were stated in the form of pure and mixed integer programming problems, in particular, and of resource allocation models, in general, that involve discrete and/or continuous variables.

This duality concept was presented for the case of two "dual" problems, where the min-max/max-min of a linear function is to be found over a domain defined by linear inequalities, and the variables are constrained to belong to arbitrary sets of real numbers, i.e. some or all of the variables are discrete.

The question of including indivisibilities in models describing economic situations were discussed, and the problems arising from such inclusion of discreteness, concerning the interpretation of results, uniqueness of optimal solutions, etc. were analyzed.

For the case of a firm operating in a purely competitive market, the linear programming concepts of shadow prices, marginal costs, free goods, and decentralization were summarized and then extended to the case of discrete programming models, using Balas's duality. It was found that the powerful linear programming concepts can be applied

also to discrete programs, by transforming them into "equivalent linear programs," if certain conditions are met.

The linearized model, however, does not constitute a means of testing optimality for the discrete program, since only a one-way relation between the problems could be established. This means that the used discrete programming algorithm to solve the original problem must provide us already with an optimal solution, before the new, linearized problem can be developed, solved, and interpreted.

Including the assumption of indivisible projects in the capital budgeting models due to Weingartner and Baumol and Quandt, and introducing the concepts mentioned above, new economic interpretations involving generalized shadow prices and a system of subsidies and penalties could be obtained for the optimal solutions to these models. One of Balas's theorems allowed the interpretation of the profit-maximizing firm in terms of a two-person zero-sum game.

More powerful results than in the case of general mixed-integer programming problems could be obtained for the pure 0-1 problem.

For Weingartner's Terminal Wealth Model, a decentralized decision making process could be developed that avoids the complete involvement of the central authority in the evaluation of the divisional investment decision problems. This finite process is a means of determining the firm's investment and financing policy, and, at the same time, internally generates the optimal set of discount rates by gradually updating them from an arbitrary initial set of feasible values.

Based on the mixed-integer versions of Weingartner's Terminal Wealth Model and Baumol and Quandt's Utility Model, a capital budgeting

model was constructed that represents a synthesis of both approaches, and the solution of which yields the optimal investment, dividend, and financing policy of a firm interacting with a non-perfect capital market. Among the issues discussed was the question of assuming finite horizon times and including post horizon cash flows, as well as the consequences of limiting the amounts of capital available for borrowing in each time period considered. Also, the relations between the optimal investment, financing, and dividend policies were discussed and some decision rules concerning these questions were obtained.

Recommendations

Some suggestions for extending this work are:

1. Include the option to carry over money from one period to the next or to a following period.
2. Use nonlinear objective functions involving both dividend payments and some equivalent of a "terminal wealth" of a firm.
3. Consider other types of (linear or nonlinear) constraints, as scarce material restrictions, space restrictions, etc. according to the particular situation considered.
4. Introduce the aspects of risk and uncertainty into the models by the use of chance-constrained programming or other appropriate methods.

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